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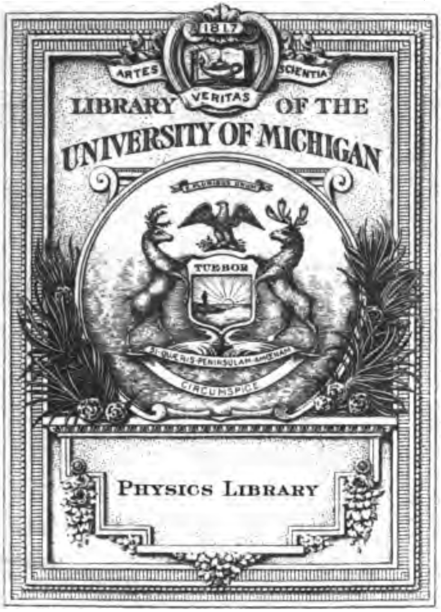

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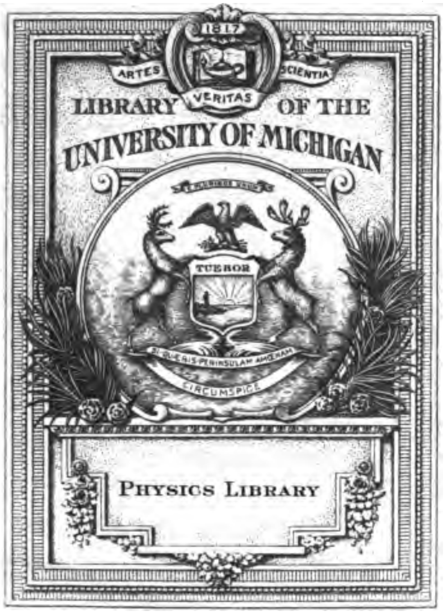

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A TREATISE

ON

SPHERICAL TRIGONOMETRY

WITH APPLICATIONS TO

SPHERICAL GEOMETRY

AND

NUMEROUS EXAMPLES

PART I.

A TREATISE

ON

SPHERICAL TRIGONOMETRY

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SPHERICAL GEOMETRY

AND

NUMEROUS EXAMPLES

BY

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PREFACE

TO THE FIRST PART.

THE object of the present Treatise is to bring Spherical Trigonometry to the standard required for University Examinations, and demanded by the impulse given to mathematical subjects by modern text-books.

Simplicity of treatment has been constantly kept in view.

Part I. treats of the subject as far as the solution of Triangles, inclusive. In the text will be found all the Propositions usually contained in treatises on the subject, besides such other Theorems as appeared to us to be of special importance on account of their utility.

The Volume is replete with examples (in many cases worked out), the arrangement of which has been the subject of our special attention: our aim throughout being to place them in immediate connexion with the subject-matter of which they are illustrative.

At the end of each Chapter, Miscellaneous Examples bearing on all the preceding matter have been added. We have not hesitated to use Determinant Notation whenever elegance or simplicity could be gained thereby. This notation has now become so generally known as to render apology for its use early in the work quite unnecessary.

The Numerical Solution of Triangles, treated of in Chapters IV. and V., has received much attention, each Case being treated of in detail. In connexion with these Numerical Examples, we must acknowledge our obligations to Mr. Robert Baile, m.a., Athlone, who carefully worked and verified them all. They have also been independently verified and tested, and we therefore trust they will all be found correct to the nearest half second.

Geometrical proofs of many Propositions have been added to those commonly given in text-books, *e. g.* the Analogies of Napier and Delambre.

Most of the examples have been taken from University and Science and Art Examination Papers. Many, however, as far as we are aware, appear now for the first time. A series of examples has been appended in the form of Examination Papers.

Among published works on the subject, we are indebted chiefly to those of Todhunter, Snowball, and Luby.

WM. J M'CLELLAND.
THOMAS PRESTON.

May, 1885.

PREFACE

TO

SECOND EDITION OF PART I.

IN the preparation of this volume the Authors have very closely adhered to the text of the First Edition. A few Propositions of fundamental importance, which previously appeared amongst the Examples, are given more prominence by being placed in the text; and some articles have been re-written and amended. The Examples are retained, with few alterations, and the Answers to them have been carefully revised.

June, 1887.

ERRATA.

Page 24, third last line, *for* "passes" *read* "pass."

" 32, lines 13 and 14, *for* "values" *read* "value."

" 58, line 2, footnote, *for* "afloat" *read* "in use."

" 61, fifth last line, *add* "(by the colunar triangle)" *after* "bisector of *C*."

" 97, fifth last line, *for* "*A* also 90°" *read* "$A = 90°$."

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CHAPTER I.

THE SPHERE.

1. **A Sphere** is a figure such that all points of its surface are equally distant from a certain point within it, called the *Centre.*

Any straight line drawn from the centre of a sphere to the surface is called a *Radius,* and any straight line drawn through the centre and terminated both ways by the surface is called a *Diameter.*

A sphere may be generated by the revolution of a semicircle round its diameter.

2. **Great and Small Circles.**—Let a sphere be generated by the revolution of the semicircle ACB (fig. 1)

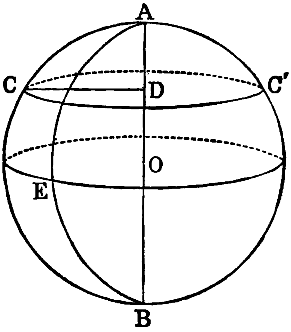

Fig. 1.

round its diameter AB. Let C be any point on the semicircle, and let CD be a perpendicular from C on AB.

It is obvious that as ACB revolves round AB, the point C describes a circle round D as centre; and also that O, the middle point of AB, being equally distant from all points on ACB, is the centre of the sphere.

When the plane of the circle CC' described by C passes through O it divides the sphere into two equal parts, and the curve of section is called a *Great Circle.*

When its plane does not pass through the centre it is called a *Small Circle.*

Example.

The Meridians and Equator are great circles. The Parallels of Latitude are small circles.

Spherical Radius.—The angular distance AC is called the *Spherical* or *Angular Radius* of the circle CC'. It is obvious that the spherical radius of a great circle is a quadrant.

3. **Unique Properties of the Great Circle.**—(1). Only *one Great Circle* can be drawn through two given points on the surface of a sphere; for its plane must also pass through the centre; and three points not in the same right line are sufficient to determine a plane completely.

If the two given points be diametrically opposite, the right line joining them passes through the centre of the sphere, and an infinite number of great circles can be drawn through them—as, for example, the meridians on the surface of the Earth.

(2). The *shortest distance* that can be traced on the surface of a sphere between two points on it is the arc of the great circle passing through them.

For of all the circular arcs which can be drawn through two given points A and B [fig. 1 (a)], obviously that arc which has the least curvature is the shortest, since it lies inside the others, and most nearly approaches the right line joining A and B. Thus the shortest arc is that which belongs to the circle of greatest radius; but the circle of greatest radius which can be drawn on a sphere is the great circle. Therefore, of all the arcs which can be drawn between two points on the surface of a sphere, the great circle arc is the shortest.*

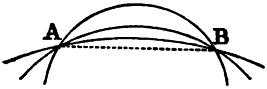

Fig. 1 (*a*).

4. **Axes and Poles.**—The line AB (fig. 1) is called the *Axis* of the circle CC' (described by *any* point C of the

---

* The following method of looking at this question is also instructive:—
If a string be stretched between two points on the surface of a sphere (or on any surface) it will evidently be the shortest distance that can be traced on the surface between the points, since by pulling the ends of the string its length, between the points, will be shortened as much as the surface will permit. Now any part of the string being acted on by two terminal tensions, and by the reaction of the surface, which is everywhere normal to it, must lie in a plane containing the normal to the surface. Hence, the plane of the string contains the normals to the surface at all points of its length; *i.e.* the string lies in the form of a great circle.

semicircle ACB during its revolution round AB), and the extremities A and B of its axis are called its *Poles.* The nearer point A is generally particularized as *the Pole* of the Circle. Any point, and the great circle of which it is the pole, are termed *pole* and *polar*, with respect to one another. It is obvious, from the manner in which CC' was generated, that AB is perpendicular to the plane of CC', for CD remained perpendicular to AB during its entire revolution.

Hence the *Axis* of a circle may be defined as *the diameter of the sphere perpendicular to the plane of the circle*, or *the line joining the centre of the sphere to the pole of the circle.*

Cor.—The pole of a circle is equidistant from all points on the circumference of the circle.

For $$AC^2 = AD^2 + CD^2 = \text{constant}.$$

In the case of a great circle, D becomes the centre of the sphere, and hence the poles of a great circle are equidistant from its circumference.

5. **Primary and Secondary Circles.**—Any circle is called a *Primary* in relation to those *great circles* which cut it at right angles. These latter are called *Secondaries;* e.g. parallels of latitude are primaries, and the meridians are secondaries to them.

In fig. 1, regarding the circle CC' as a primary, the circle ACB during its revolution round AB is in all positions a secondary to CC'. Hence it follows that—

(1). The plane of any secondary contains the axis of the primary.

(2). All the secondaries pass through the poles of the primary.

(3). The planes of all the secondaries have a common line of intersection, viz. the axis of the primary.

(4). If there can be drawn common secondaries to two circles, the planes of those circles are parallel. For, by (3), the two circles have the same axis.

The distance of any point on the surface of a sphere from a circle traced thereon is measured by the arc of the secondary intercepted between the point and the circle.

Example.

The latitude of a place is measured by the arc of the meridian intercepted between the place and the Equator.

6. **The Angle between Two Planes** is measured by the angle between any two lines drawn, one in each plane, perpendicular to their line of intersection.

The angle between any two circles, great or small, is measured by the angle between the tangents drawn to them at their point of intersection.

The Angle between Two Great Circles is equal to—

(1). *The angle between their planes.*

For the tangents to them at their point of intersection lie in their respective planes and are perpendicular to their common diameter.

(2). *The arc intercepted by them on the great circle to which they are secondaries.*

For if O be the centre of a sphere, and AC and BC (fig. 2) secondaries to AB, they are perpendicular to AB, and are each quadrants; therefore OA and OB are perpendiculars to OC; hence the angle between OA and OB, or the arc AB, is the angle between the planes of AC and BC.

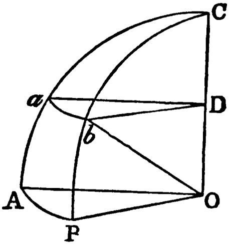

Fig. 2.

(3). *The angular distance between their poles.*

For the angular distance between the poles of two circles is the same as the angle between their axes; but the axes are perpendicular to the planes of the circles, and the angle between the perpendiculars to two planes is equal to the angle between the planes.

Hence, by drawing a great circle parallel to a small circle, we see that the angle between the planes of a great and a small circle, or the angle between the planes of two small circles, is equal to the angular distance between the poles of the circles.

7. *To compare the length of the arc of a small circle subtending any angle at its centre with the arc of a great circle subtending an equal angle at the centre of the sphere..*

Let ab (fig. 2) be an arc of a small circle subtending any angle at its centre D, and let AB be an arc of a great circle subtending an equal angle at its centre O, then if the circles be placed parallel, so that they may have a common pole at C, O will be the centre of the sphere, AC and BC will be quadrants, CO will be perpendicular to the planes ABO and abD; and since the perimeters of circles are to each other as their radii, we have

$$\frac{ab}{AB} = \frac{bD}{BO} = \frac{bD}{bO} = \sin bOD = \sin bC = \cos bB.$$

The above equation expresses the length of the arc of a parallel of latitude in terms of the latitude of the place and the corresponding arc of the Equator. Thus $ab = AB$ cos (latitude); or, *the distance* (ab) *between two places in the same latitude, measured on the parallel, is equal to the difference of longitude* (AB) *multiplied by the cosine of the latitude.*

8. **The Lune.**—A *Lune* is a portion of the surface of a sphere enclosed by two great circles.

In fig. 1 the area $ACBEA$ is a lune; so also is the area $AEBC'A$.

The angle between the two great circles bounding the lune is called the *Angle of the Lune.*

The area of a lune is readily expressed in terms of the angle of the lune and radius of the sphere. For we have evidently—

Area of Lune : Area of Sphere : : Angle of Lune : 2π; or, denoting the angle of the lune by A, and assuming* the area of the sphere to be $4\pi r^2$, where r is the radius of the sphere,

---

* See Part II., Art. 97.

$$\text{Area of Lune} = 2Ar^2.$$

9. **Theorem.**—*Any section of the surface of a sphere by a plane is a circle.*

For (fig. 1) if OD be let fall from O, the centre of the sphere, perpendicular to the plane CC', we have

$$CD^2 = CO^2 - OD^2 = \text{constant},$$

since CO is constant, and OD is fixed. Hence, every point of the section of the surface is equidistant from D, and therefore lies on a circle round D as centre.

10. *The intersection of two spherical surfaces is a circle,*

Consider the triangle ABX formed by joining the centres A and B of the two spheres and any point X on their curve of intersection. The sides of this triangle are *given*, and the line of centres AB is fixed; hence the opposite vertex X (*i.e.* the variable point on the curve of section of the two spheres) is at a constant distance from AB, or lies on a circle having AB for axis.

This property is more general than that stated in the previous Article, and reduces to it as a particular case by supposing the radius of either sphere to become infinitely great.

11. **Solid (or Conical) Angles.**—When three or more planes intersect at a point they enclose what is termed a *Solid Angle* at that point. The much more expressive term, *Conical Angle*, is also used. There is a solid angle, for example, at each corner of a cube, and at each

vertex of a tetrahedron or pyramid. Every closed curve, plane or tortuous, subtends a solid angle at any point in its neighbourhood, the solid angle being enclosed by the lines drawn from the point to every point of the contour of the curve.

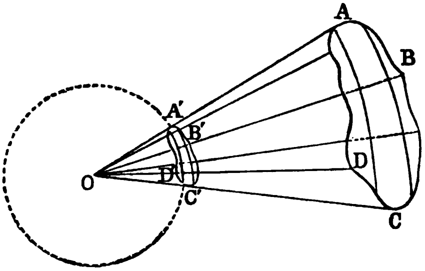

Fig. 3.

Let *ABCD* (fig. 3) be any curve or plane area. From any point *O* draw an infinite number of lines *OA*, *OB*, *OC*, *OD*, &c., passing through the boundary of *ABCD*. These lines form an irregular cone, having its vertex at *O*, and enclosing the curve or area *ABCD*. The solid angle of this cone is the solid angle which *ABCD* subtends at *O*.

To measure this angle, an expedient similar to that employed in reckoning plane angles by circular measure is used. About *O* describe a sphere of unit radius, and let the cone formed by the lines from *O* to *ABCD* intersect the surface of this sphere in the curve *A′B′C′D′*. The curve *A′B′C′D′* will enclose an area on the sphere, and this area measures the solid angle at *O*, just as the arc of a circle of unit radius measures the angle it subtends at the centre. The student must be careful, however, not to regard a solid angle as an *area*, but as a mere *number*, like the circular measure of a plane angle. For as the circular measure of a plane angle is the ratio of the length of the arc subtending it to the radius of the circle, so the ratio of the area $A'B'C'D'$ to the square of the radius of the sphere is the true measure of the solid or conical angle. When the radius of the sphere is unity this ratio becomes numerically equal to the number of units of area enclosed by $A'B'C'D'$.

Since the radius of the sphere is unity, its area is 4π; hence the sum of all the solid angles round any point is 4π, and the solid angle subtended at O by the $1/n^{th}$ part of the surface of the sphere is $4\pi/n^{th}$, which is thus a mere number. For example, the solid angle contained by three mutually rectangular planes, as at a corner of a cube, is $\frac{1}{2}\pi$.

Examples.

1. If O is situated inside a closed surface, the sum of the solid angles subtended at O by all the elements of the surface is 4π.

2. If in Ex. 1 O is situated on the surface, the sum of all solid angles is 2π. [For in this case the lines joining O to all the points of the surface intercept half the area of the sphere.]

3. As a case of Ex. 2, the solid angle subtended at any point by an infinite plane, or by any plane curve at a point enclosed by it, is 2π.

4. The solid angle of a right cone of semi-vertical angle α is $2\pi(1-\cos\alpha)$. [*See* Art. 97.] Thus the solid angle subtended by a circle of radius r at a point on its axis, distant z from the plane of the circle, is $2\pi\left(1-\frac{z}{\sqrt{r^2+z^2}}\right)$.

12. **Theorem.**—*The point of intersection of two great circles and the arc joining their poles are pole and polar with respect to one another.*

For let A and B be two points, LO and MO their polars, O the point of intersection of the polars; then $AO = \frac{1}{2}\pi$, $OB = \frac{1}{2}\pi$; hence OA and OB are secondaries to AB; therefore, &c. [Art. 5 (2).]

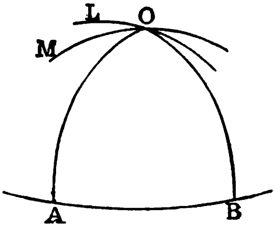

Fig. 4.

Examples.

1. Two great circles bisect each other.

They have a common diameter.

2. The axis of any circle is determined by the intersection of two of its secondaries. [Art. 5 (3).]

3. Find the locus of the centres of a system of circles having common secondaries. *Ans.* The common axis of the system.

4. The pole of any great circle is ninety degrees from the circumference. [Art. 4.]

5. The arc drawn through the poles of two great circles cuts both at right angles.

5*a*. Draw a great circle cutting two given circles, great or small, at right angles.

6. If three or more great circles are concurrent, their poles are concyclic.

7. Any great circle is the locus of the poles of all its secondaries.

8. If from any point on the surface of a sphere *two* great circles can be drawn perpendicular to a given great circle, that point is the pole of the circle.

[For, each of the arcs drawn perpendicular are secondaries; therefore, &c. Art. 5, (2).]

9. The external bisector of an angle passes through the pole of the internal bisector; and conversely.

10. If two great circles are equally inclined to a third, their poles are equidistant from the pole of the third.

10*a*. If a point is equidistant from three great circles, it is also equidistant from their poles.

11. Two equal small circles are drawn touching each other: show that the angle between their planes is twice the complement of their spherical radius.
—(*Science and Art Examination Papers.*)

[For, join the centres of the small circles to the centre of the sphere, and also to the point of contact of the circles.

Then the lines drawn to the centre of the sphere contain an angle equal to twice the spherical radius of either small circle, and the lines drawn to the point of contact of the small circles contain an angle equal to the inclination of the planes of the circles; therefore, &c. —(Euc. III. XXII.)]

12. On a sphere whose radius is r a small circle of spherical radius, θ, is described, and a great circle is described having its pole on the small circle; show that the length of their common chord is

$$\frac{2r}{\sin\theta}\sqrt{-\cos 2\theta}.$$

—(*Science and Art Examination Papers.*)

[For the angle between the axis of the small circle and the plane of the great circle is evidently $90 - \theta$. Hence the distance, d, from the middle point of the common chord to the centre of the small circle is given by the equation

$$d = r\cos\theta\cot\theta.$$

Again, if c be the common chord, it follows that

$$\left(\frac{c}{2}\right)^2 = (r\sin\theta + d)(r\sin\theta - d),$$

since $r \sin \theta$ is the radius of the small circle; therefore

$$c = \frac{2r}{\sin \theta} \sqrt{- \cos 2\theta},$$

the negative sign occurring under the radical, since 2θ must be greater than 90° for *real* section.]

Note—Hence the segments of a great circle, made by the intersection of a small circle of given radius, may be calculated on a sphere of given radius.

13. The angle subtended at the centre of a circle, great or small, by two points on it, is equal to the angle subtended by them at its pole. [Art. 6.]

14. If through the centre O of a sphere a line be drawn parallel to a chord AB of a semicircle ABC described thereon, it meets the sphere at the middle point of the arc BC.

15. Given A, B, C, D, four points on a segment of a circle; prove the relations—

$$\alpha^\circ. \quad \sin BC \sin AD + \sin CA \sin BD + \sin AB \sin CD = 0.$$

$$\beta^\circ. \quad \sin BC \cos AD + \sin CA \cos BD + \sin AB \cos CD = 0.$$

$$\gamma^\circ. \quad \sin \frac{BC}{n} \sin \frac{AD}{n} + \sin \frac{CA}{n} \sin \frac{BD}{n} + \sin \frac{AB}{n} \sin \frac{CD}{n} = 0.$$

CHAPTER II.

SPHERICAL TRIANGLES.

13. In the present chapter it will be shown that geometrical figures drawn on the surface of a sphere (or different spheres of equal radii) have many properties and relationships analogous to those of corresponding figures described on a plane surface; and with the latter the reader is supposed to be already familiar.

14. **Definitions.**—By a *Spherical Figure* is meant a portion of the surface of a sphere enclosed by arcs of *great** circles, *e.g.*

A *Spherical Triangle* is bounded by the arcs of three great circles; a *Spherical Quadrilateral* by four great circles; a *Spherical Polygon* by many great circles.

In the particular case, when the number of enclosing arcs becomes indefinitely great, the figure becomes a *Spherical Curve.* A great circle drawn through two indefinitely near points on a curve is a *Tangent Circle* to the curve. Great and small circles are examples of spherical curves.

The arcs are generally spoken of as the *Sides*, and their angles of inclination as the *Angles* of the spherical figure.

Since, however, three great circles on a sphere intersect one another so as to form *eight* triangles, for the sake of convenience, and to avoid ambiguity, that particular triangle is selected which has two, or if possible three, sides, each less than a quadrant.

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* Unless otherwise stated.

The three sides and three angles are termed the *Parts* of a spherical triangle; and, as will be hereafter seen, any three whatever given in magnitude are necessary and sufficient to determine the remaining three.

15. **By the Arc connecting two Points** on a sphere, unless otherwise expressed, we mean the lesser segment of the great circle passing through the points. (See Chap. I. Art. 3.)

Examples.

1. Each side of a spherical triangle is less than two right angles.

2. Each angle of a spherical triangle is less than two right angles.

[For if possible let ABC be a spherical triangle, having an angle at A greater than two right angles. Continue the arc BA to the point X on the side BC.

Then BX is a semicircle; hence $BX + CX$ is greater; therefore ABC is not a spherical triangle.]

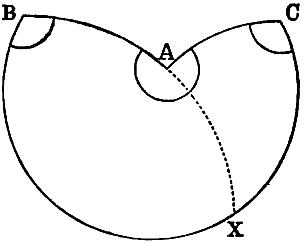

Fig. 5.

3. The area of any spherical triangle is less than $2\pi r^2$. [See Art. 8.]

16. **Colunar or Associated Triangles.**—By producing the sides of a triangle to meet below the base, a triangle is formed, having two sides and the opposite angles respectively supplemental to two sides and the opposite angles of the original triangle, while the remain-

ing angle of the one is equal to the remaining angle of the other (being opposite angles of a lune). Triangles whose parts are so related are termed *Colunar*, and the three triangles colunar with the given one are sometimes termed its *Associated Triangles.* The triangle taken in the first instance is called the *Primitive Triangle.*

Examples.

1. The three associated triangles, taken in pairs, have a side and an opposite angle of the one equal to a side and an opposite angle of the other.

2. When a triangle is equilateral, the colunars are each isosceles.

3. The arcs joining the vertices of the colunar triangles are equal to the sides of the primitive triangle.

17. **Parts of a Spherical Triangle.**—Let ABC be a spherical triangle described on a sphere having its centre at O. Then $OA = OB = OC =$ the radius of the sphere. The arcs BC, CA, and AB are proportional to the angles BOC, COA, and AOB, which they subtend at the centre, and the angles BAC, ABC, and BCA between the sides are equal to the angles between the planes containing the solid angle at O. (Chap. I., Art. 6.)

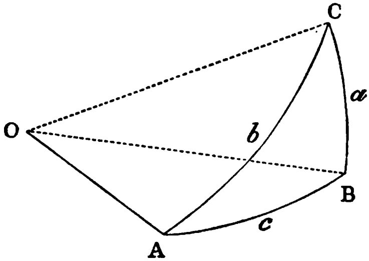

Fig. 6.

The notation for the parts of a spherical triangle adopted throughout will be that of Fig. 6, viz. for the three angles the letters A, B, and C, and for the opposite sides the small letters a, b, and c, respectively; though, as will be hereafter noticed, the letters A, B, and C, may denote indifferently the number of degrees in the angles or the circular measure of the angles.

Any expression involving one or more parts of a triangle is a *Function* of these parts, and may be represented in a general form such as—for the side $a, f(a)$; for the sides a, b, and $c, f(a, b, c)$; for two sides b and c, and the angle $A, f(b, c, A)$; for all the parts, $f(a, b, c, A, B, C)$.

18. **Sphere of Infinite Radius.**—At the point of intersection of two great circles, let two tangents be drawn to them in their respective planes; the plane containing both tangents is, with respect to the sphere, the *Tangent Plane* at the point. Now consider the radius of the sphere to increase indefinitely. It is manifest that as the radius increases, each element of surface, and, as a consequence, the whole surface in the neighbourhood of the point, approaches nearer and nearer to the plane; and the great circles described thereon approach nearer and nearer to the tangents. In the limiting case, when the radius of the sphere becomes indefinitely great, and the curvature therefore indefinitely small, the great circles approximate so closely in position to the tangents that they may be regarded as coincident with one another.

Thus the sphere and the great circles degenerate respectively into two* planes (one of which is at infinity), and two pairs of lines (one pair being also at infinity).† But it is to be observed that the angle of inclination of the planes of the great circles alone remains unaltered (*cf.* Chap. I., Art. 6).

Hence the relations involving certain parts of a spherical triangle may be regarded as more general than those involving similar parts of a plane triangle; and by the aid of the expansions

$$\sin a = \frac{a}{r} - \frac{1}{\underline{|3}}\frac{a^3}{r^3} + \frac{1}{\underline{|5}}\frac{a^5}{r^5} - \ldots \text{ to infinity,} \qquad (1)$$

and

$$\cos a = 1 - \frac{1}{\underline{|2}}\frac{a^2}{r^2} + \frac{1}{\underline{|4}}\frac{a^4}{r^4} - \ldots \text{ to infinity} \qquad (2)$$

(see *Plane Trigonometry*), any function of A, B, C, a, b, c, the parts of the former, can be transformed into a function of A, B, C, α, β, γ, the parts of the latter. It should, however, be clearly understood, that α represents the *length* of an arc subtending an angle a, at the centre of

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* In accordance with analytical geometry; but it is quite sufficient to understand that any finite portion of the surface is a plane.

† Reference to the Earth, considered as a large sphere, will elucidate this statement.

The surface of a lake or any small portion of water is apparently a perfect plane, although it partakes of the general curvature of the Earth, which may be observed in the case of the ocean, where a distinct curvature is exhibited. It can therefore be readily imagined that if the radius of the Earth were greatly increased, even the surface of the ocean would present no perceptible curvature.

Now take any point on the surface, and draw a plane passing through it a sphere of radius r; and when r becomes infinite the arc a becomes a right line, viz. a side of the corresponding plane triangle.

It thus appears that every great circle forming part of a diagram in Spherical Geometry will be represented by a right line in the corresponding diagram *in plano;* but we must carefully remember that every right line of a plane figure has not necessarily a representative great circle in the corresponding figure on the sphere; for it will appear hereafter that in some cases small circles on the sphere also become right lines *in plano.*

For example, it is well known that when the base and area of a plane triangle are given, the locus of its vertex is a right line parallel to the base; but when the base and area of a spherical triangle are given, the locus of its vertex is a *small circle* passing through the points diametrically opposite to the extremities of the given base. (See Art. 101.) A moment's consideration will make it plain that the small circle locus here mentioned should *in plano* be represented by a right line, and still further

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and through the centre of the Earth. This plane will cut the surface of the Earth in a great circle (Chap. I., Art. 9). But the surface, at any point, being sensibly a plane, the trace of this great circle on it will be approximately a right line, since the intersection of two planes is a right line. If another great circle be drawn through the point, the two will not differ sensibly from a pair of right lines intersecting at it; and since they both pass through the diametrically opposite point, they will intersect at it also as a pair of right lines, the surface of the Earth being at this point also approximately a plane. Now if the radius of the Earth were indefinitely increased, no curvature whatever would be observed at any part of its surface, and the point diametrically opposite to any selected one would be infinitely distant from it, and the results stated above would follow.

a right line parallel to the base of the triangle. For it passes through the points diametrically opposite to the extremities of the base, and *in plano* these points are infinitely distant; hence the small circle in question becomes one of infinite radius, that is, a right line; and it must meet the given base at infinitely distant points, that is, it must be parallel to it. Hence we are not to conclude that all rectilinear loci in Plane Geometry are represented by great circle loci on the sphere; but, on the contrary, that any small circle on the sphere, constrained to pass through a point diametrically opposite to any point of a spherical figure, is represented by a right line in the corresponding plane figure.

Another example illustrating this principle will be found in Art. 169, *Cor.* 6.

19. **The Analogy between Problems in Plane and Spherical Trigonometry** referred to in the previous Article may be more clearly illustrated as follows:—

(1). *Any two sides of a triangle are together greater than the third side.*

[Proof as *in plano*, *v.* Euc. I. xx. Otherwise thus:— The great circle AB (fig. 6) is the shortest distance measured on the sphere between the points A and B; therefore

$$AC + BC > AB. \qquad \text{(See also Euc., XI. xx.)}]$$

(2). *Only one triangle can be constructed with three given arcs* (*any two of which are greater than the third*).

[If we attempt to construct a triangle with three given arcs, as in Euc., I. xxii., we obtain two triangles on the same base with their vertices on opposite sides of it. These triangles cannot be shown equal by direct superposition* on account of the curvature of their surfaces, but they may be divided into others which can be superposed (see Note, Art. 100); and it is also evident that one of them can be placed on the sphere with its vertices diametrically opposite to the vertices of the other. In this position it is clear that the angles of one are equal to the angles of the other; for their sides are parts of the same three great circles, and their angles are the angles between the planes of these circles, or the angles of equal lunes. Two such triangles are represented in fig. 7, and are said to be *symmetrically equal.*

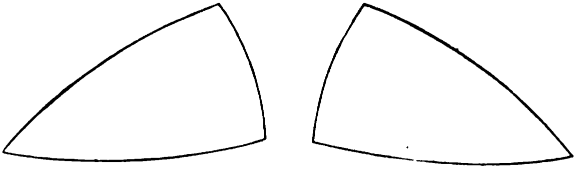

Fig. 7.

(3). *If two triangles have two sides, and the included angle of one respectively equal to the two sides and the included angle of the other, the triangles are equal in every respect.*

[Euc., I. iv.]

[For two sides of given length intersecting at a given angle determine the vertices of the triangle. Therefore (Art. 3 (1)) the base and remaining angles are determined, and the triangle is unique.]

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* If the triangles be cut out from a thin flexible spherical shell, they can be superposed directly by reversing the curvature of one of them.

(4). *The angles at the base of an isosceles triangle are equal to one another.*[*]

[Proceed exactly as in Euc., I. v., using (3)].

(5). *If two angles of a triangle be equal, the triangle is isosceles.*

(6). *The sum of one pair of opposite angles of a quadrilateral inscribed in a circle is equal to the sum of the remaining pair.*[†]

[For by joining the pole of the circle to the angles of the quadrilateral we have four isosceles triangles; therefore, &c., by (4).]

(7). *The sum of the sides of a quadrilateral is greater than the sum of the diagonals.*

[Apply (1).]

(8). *The greater side of every triangle is subtended by the greater angle.*

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[*] Otherwise thus:—Let ABC be an isosceles triangle on a sphere, with centre O. At A and B draw tangents to the equal sides CA and CB, in their respective planes. These tangents will intersect OC, produced, in the same point T. For if possible let the tangent at B intersect it in T'. Then in the two triangles AOT and BOT' we have the angles TOA and $T'OB$ equal, the angles OAT and OBT' right angles, and $OA = OB$; therefore (Euc., I. xxvi.) $OT = OT'$.

Also $AT = BT$, and the tangents to AB at its extremities are also equal; therefore, &c., Euc. I. viii. This being established independently, (2) and (3) follow from it.

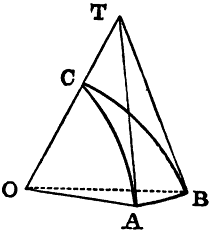

Fig. 6 (*a*).

[†] This property may be regarded as the criterion of a cyclic spherical quadrilateral. (See Art. 25*a*.)

For, let ABC (fig. 8) be the given triangle, having the angle B greater than the angle A. Draw a great circle BX, making the angle ABX = angle A. Then $BX = AX$. But $BX + CX > BC$; therefore $AC > BC$.

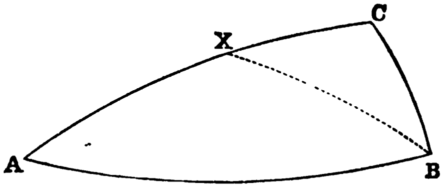

Fig. 8.

It may be easily shown that the theorems given in Euc. I., Props. XV., XVIII., XXI., XXIV., XXV., XXVI., are more generally true for the sphere.

(9). *The internal bisectors of the angles of a triangle, or two external and one internal, are concurrent.* [(Euc. IV. v.)]
[The points of concurrence are the poles of the small circles touching the sides of the triangle.]

(10). *The arcs of great circles bisecting the sides of a triangle at right angles are concurrent.* [(Euc. IV. v.)]
[The point of concurrence is the pole of a small circle passing through the vertices of the triangle.]

(11). *Given the base and sum of base angles, the external bisector of the vertical angle always passes through a fixed point.* (Dublin University Exam. Papers.)

For, let ABC (fig. 9) be a triangle having the given base AB. Upon AB construct an isosceles triangle AOB, having each of the base angles OAB and $OBA = \frac{1}{2}(A+B)$.

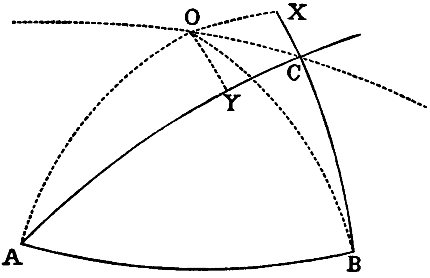

Fig. 9.

Draw OX and OY secondaries to BC and AC, respectively.

In the two right-angled triangles BOX and AOY, $AO = BO$, $\angle\, OAY = \angle\, OBX = \frac{1}{2}(B-A)$; therefore $AY = BX$, and $OX = OY$.

Again, compare triangles COX and COY: $OX = OY$, OC is common, and the angles at X and Y are right angles; hence CO is the bisector of the external vertical angle passing through the fixed point O above determined.

(12). Given of a triangle the base and *sum* or difference of base angles, the *internal* and *external bisectors* of the vertical angle passes through fixed points.

[Cf. Prop. (11).]

20. **Theorem.**—*The great circle bisecting the sides of a triangle intersects the base* 90° *distant from its middle point.*

Let ABC (fig. 10) be the triangle; X, Y, Z, the middle points of the sides; AL, BM, and CN, secondaries drawn through the vertices of the triangle to the great circle XY, passing through the middle points of the sides. Let XY meet the base in the points D and E, which are, therefore, diametrically opposite. O is the pole of XY. (Chap. I. Ex. 8.)

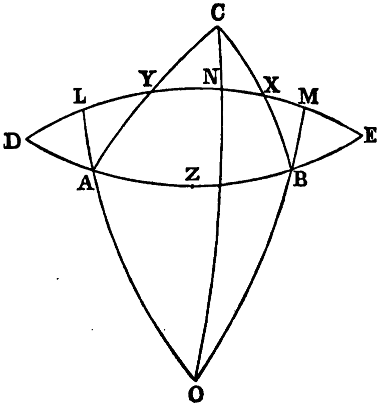

Fig. 10.

Now in the two triangles ALY and CYN we have $AY = CY$, $\angle AYL = \angle CYN$, and $\angle ALY = \angle CNY$; therefore the triangles are equal in every respect. Hence $AL = CN$. Similarly it can be shown that $BM = CN$.

Therefore

$$AL = BM = CN.$$

Again, in the two triangles ALD and BME it follows easily that $DL = ME$ and $AD = BE$; hence

$$DZ = ZE = 90°.$$

Cor.—The perpendiculars of a Spherical Triangle meet in a point.[*]

[Regarding XYZ (fig. 10) as the triangle, the perpendicular from Z on XY is, by the above, the polar of the point E. It is, therefore, perpendicular to AB. Similarly the perpendiculars from X and Y on the sides YZ and ZX are at right angles to BC and CA, respectively; therefore, &c., Art. 19 (10)].

Exercises on Fig. 10.

1. The arcs DL and XY are complementary.
2. The triangle AOB is isosceles.
3. The angle $LAY = S - A$, where $2S = A + B + C$.
4. The arcs OC and OA are supplementary.
5. Given the base and the sum of the three angles, the locus of the vertex C is a small circle having its pole at O.

[For under the given conditions the triangle AOB is fixed; hence also OC, by the aid of Ex. 4; therefore, &c.]

6. D and E are the middle points of the sides of the colunars on AC and BC respectively.

Polar Triangles.

Definition.—Two triangles so related that the vertices of the one are the poles of the sides of the other are called, with respect to one another, *Polar Triangles.*

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[*] The proof here given is due to Sadlier. See also Art. 54.

21. **Reciprocal Relations of the Sides and Angles of Polar Triangles.**—*If two triangles be so related that the vertices of the one are the poles of the corresponding sides of the other; then, conversely, the vertices of the latter shall be the poles of the corresponding sides of the former.*

Let the points A, B, and C (fig. 11), be respectively the poles* of $B'C'$, $C'A'$, and $A'B'$ the sides of the triangle $A'B'C'$. Produce the side $B'C'$ (if necessary) to meet AC in the point M, and AB in the point N.

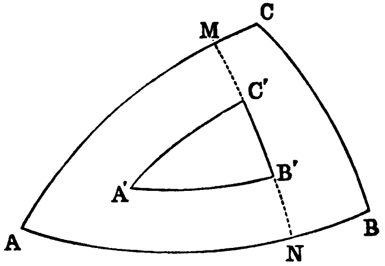

Fig. 11.

Now, since A is the pole of $B'C'$, the arc AB' is a quadrant; and since C is the pole of $A'B'$, the arc CB' is a quadrant; hence B' is the pole of AC: similarly, A' is the pole of BC, and C' the pole of AB; therefore, &c.

[Art. 12.]

Cor. 1.—The arcs connecting the corresponding vertices of a triangle and its polar pass through a point, viz. the common *Orthocentre* of the two triangles.

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* The poles of the sides of a triangle may be conveniently obtained by drawing great circles through the vertices perpendicular to the opposite sides, and by taking points on them (measured from the sides towards the vertices) 90° distant from the sides. These points are evidently the required poles.

[For the arc joining A and A' (fig. 11) is perpendicular to each of the arcs BC and $B'C'$, since it passes through their poles; therefore, &c., Art. 20, *Cor.*].

Cor. 2.—The sides of a triangle intersect the corresponding sides of the polar triangle in six points lying on a great circle.

[For the arc AA' is the polar of the points of intersection of BC and $B'C'$: similarly the arcs BB' and CC' are the polars of the points of intersection of CA, $C'A'$ and AB, $A'B'$; therefore, &c., by *Cor.* 1.]

22. **Theorem.**—*The angles of the primitive triangle are the supplements of the corresponding sides of the polar; and conversely.*

Since B' (fig. 11) is the pole of AC, $B'M$ is a quadrant, and for a similar reason $C'N$ is also a quadrant (Chap. I., Ex. 4).

Hence $MN + B'C' =$ two right angles.

But MN is equal to the angle A; therefore the angle A and the side $B'C'$ are supplemental.

Similarly, the angles B and C are the supplements of the sides $C'A'$ and $A'B'$ respectively; therefore, &c.

Note.—From the above property, Polar Triangles are also termed *Supplemental Triangles;* and if a', b', c', A', B', C', represent the parts of the triangle $A'B'C'$ (fig. 11), it follows that

$$A + a' = B + b' = C + c' = \pi.$$

Also

$$A' + a = B' + b = C' + c = \pi.$$

These results are of the greatest importance, inasmuch as any theorem which holds good between the sides and angles (e.g. a, b, c, A, B) of a spherical triangle necessarily involves a reciprocal or supplemental theorem, involving the opposite angles and sides (viz. A, B, C, a, b), and which may be derived from it by changing the sides into the supplements of the corresponding angles, and the angles into the supplements of the corresponding sides.

23. The results obtained in the previous Article may be further exemplified, as follows:—

(a). It has been seen that two sides of a triangle are greater than the third: thus $b + c > a$.

Applying this inequality to the sides of the supplemental triangle, it follows that

$$\pi - B + \pi - C > \pi - A.$$

Hence, $\quad B + C - A < \pi$, i.e. $S - A < \frac{\pi}{2}$. $\quad$ (1)

Similarly, $\quad C + A - B < \pi$, i.e. $S - B < \frac{\pi}{2}$. $\quad$ (2)

And $\quad A + B - C < \pi$, i.e. $S - C < \frac{\pi}{2}$. $\quad$ (3)

Or, *the difference between any two angles of a spherical triangle is less than the supplement of the third angle.*

Particular cases of the inequalities (1), (2), and (3), are worthy of notice.

a. Suppose, for example, that the primitive triangle is right-angled at C, it follows from (3) that

$$A + B < \frac{3\pi}{2};$$

or, *the sum of the angles of a right-angled triangle is less than four right angles.*

β. It likewise follows, by the aid of (2), that

$$A - B < \frac{\pi}{2};$$

or, *the difference of the oblique angles of a right-angled triangle is less than a right angle.*

(*b*). Prop. (4), Art. 19, involves the converse theorem, viz., If two angles of a triangle are equal, the sides subtending them are also equal. [See Prop. (3), Art. 19.]

(*c*). Prop. (8), Art. 19, involves the converse theorem, viz., The greater angle is subtended by the greater side.

(*d*). Prop. (3), Art. 19, involves the supplemental theorem, viz., If two triangles have two angles and a side of the one equal to two angles and a side of the other, the sides being similarly situated with respect to the equal angles, the triangles are equal in every respect.

(*e*). In any triangle, if $C = A + B$, then $C - A$ and $C - B$ are each less than a right angle.

For $C - A = B$, and $C - B = A$; but $B + C - A < \pi$; hence $B < \frac{\pi}{2}$, and $A < \frac{\pi}{2}$; therefore, &c.

Examples.

1. Given two angles of a triangle to be 45° and 120°, find the maximum value of the third angle.

Let A be the third angle, then by aid of (1),

$$A + 120 - 45 < 180°.$$

Hence,

$$A < 105°.$$

2. The angles of a triangle are A, 30° and 150°; find the maximum value of A. *Ans.* $A < 60°$.

[Proof as before.]

3. If the angles are A, 20° and 110°, find the maximum value of A. *Ans.* $A < 90°$.

4. If the difference between any two angles of a triangle is equal to 90°, the remaining angle is less than 90°. [Cf. Art. 23, β.]

5. If the primitive triangle be equilateral or isosceles, the supplemental triangle is equilateral or isosceles.

24. **Theorem.**—*If two sides of a triangle are supplemental, the opposite angles are supplemental.*

For in the triangle ABC (fig. 12), if $AC + AB = \pi$,

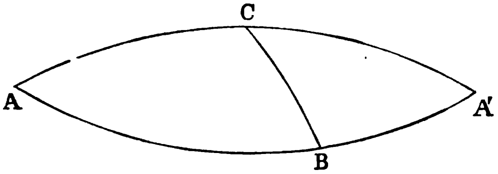

Fig. 12.

since the arc ABA' is a semicircle, then will

$$AC = A'B, \text{ and } A'C = AB.$$

Therefore the two triangles ABC and $A'BC$ are equal in every respect; therefore, &c.

Examples.

1. If two sides of a triangle are supplemental, two sides of the polar triangle are likewise supplemental.

2. In fig. 12, the arc joining the middle point M of BC with A or A' is a quadrant.

[The triangles $A'BM$ and ACM are equal in every respect.]

3. Given the base of a spherical triangle, and the sum of the sides equal to two right angles, find the locus of the vertex.

Ans. A great circle, having the middle point of the base as pole (Ex. 2).

25. **Angular Limits of the Sides and Angles of a Spherical Triangle.**—It has been said that each side and each angle of a spherical triangle may have any values between 0° and π. It is, therefore, manifest that the sum of the angles cannot exceed 3π; and it will be here shown that the sum of the sides cannot exceed 2π. For in the triangle $A'BC$ (fig. 12),

$$BC < A'C + A'B.$$

To each add

$$AC + AB.$$

Hence,

$$BC + CA + AB < AC + A'C + AB + A'B < 2\pi.$$

Hence $a + b + c$ *can have any values between* 0° *and* 2π,

and $A + B + C$ *can have any values between* π *and* 3π.

The latter may be seen otherwise, thus:—

Since 0°, $a + b + c$, 2π, are in ascending order of magnitude; hence, for the supplemental triangle,

$$0°,\quad \pi - A + \pi - B + \pi - C,\quad 2\pi,$$

are likewise in ascending order of magnitude; therefore, &c.

25 (a). **Theorem.**—*If the sum of one pair of opposite angles of a spherical quadrilateral be equal to the sum of the other pair, the quadrilateral is cyclic.*

For let $ABCD$ [fig. 12 (a)] be a quadrilateral, such that the sum of the angles A and C is equal to the sum of the angles B and D; and if possible let the circle passing through B, C, D not pass through A, but let it, as in

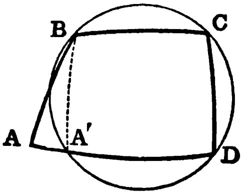

Fig. 12 (*a*).

Fig. 12 (*a*), intersect the arc AD in some point A'. Join BA'.

Then, by Art. 19 (6), since $A'BCD$ is in a circle,

$$A' + C = D + A'BC.$$

But

$$A + C = D + ABC.$$

Therefore, by subtracting,

$$A - A' = ABA', \quad \text{or } A - ABA' = A',$$

which is impossible, since A' is the external angle of the triangle ABA'; and, by Art. 23 (*a*), the difference of any two angles of a triangle is less than the supplement of the third. Therefore the points A, B, C, D lie on the same circle, *i.e.* the quadrilateral is cyclic.

25 (*b*). **Analogue of Ptolemy's Theorem.**—*If a, b, c, d be in order the sides, δ and δ' the diagonals of a spherical quadrilateral inscribed in a circle; then*

$$\sin \tfrac{1}{2}a \sin \tfrac{1}{2}c + \sin \tfrac{1}{2}b \sin \tfrac{1}{2}d = \sin \tfrac{1}{2}\delta \sin \tfrac{1}{2}\delta'.$$

For if a', b', c', d' be the chords of the arcs which form the sides of the quadrilateral, e' and f' those of the diagonals, it is clear that a', b', &c., are the sides and diagonals of a plane quadrilateral inscribed in a circle, and therefore

$$a'c' + b'd' = e'f';$$

but $a' = 2r \sin \frac{1}{2}a$, $b' = 2r \sin \frac{1}{2}b$, &c., where r is the radius of the sphere. Therefore, &c.

Examples.

1. Any side of a triangle is greater than the difference between the other two sides.

2. When does the primitive triangle coincide with the supplemental?

Ans. When its sides and angles are each $\frac{\pi}{2}$.

3. Assuming the earth to be a sphere, is the area included between two meridians and a parallel of latitude a spherical triangle? Give a reason for your answer. —(*Science and Art Exam. Papers.*)

4. The sides b and c of a triangle are produced both ways to points x and x', y and y', 90° distant from the middle point of the sides. Show that, if secondaries to the sides from x and y intersect in P, and from x' and y' in P', the points P and P' are diametrically opposite.

5. The triangle $P'Q'R'$ is the polar triangle of that formed by joining the middle points of the sides of ABC, where Q' and R' are formed by similarly producing the sides c and a, a and b, through the extremities of the remaining side.

NOTE on Art. 26.

[The formula of Art. 26 may be obtained directly by equating the projection of OD on OE to the sum of the projections of OC aud CD on the same line. Thus

$$OD \cos c = OC \cos b + CD \cos C \sin b,$$

or

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.]$$

CHAPTER III.

FUNDAMENTAL RELATIONS BETWEEN THE PARTS OF A SPHERICAL TRIANGLE.

Section I.

26. *Having given the sides of a spherical triangle, to determine the cosines of the angles.*

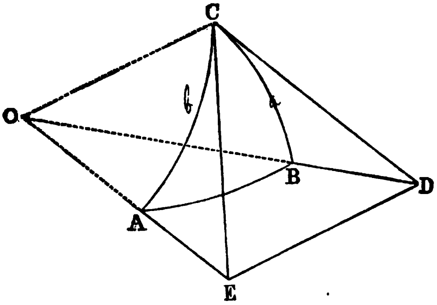

Fig. 13.

Let ABC be a spherical triangle; a, b, and c, its sides; O the centre of the sphere. Draw CD and CE tangents to a and b at C.*

Since these tangents lie in the planes of the circles to which they are drawn, they will meet the radii OB and

---

* The same result will be obtained if, instead of drawing tangents at C, any point C' be taken on OC, and lines $C'D'$ and $C'E'$ be drawn perpendiclar to it, meeting OB and OA in D' and E', respectively.

OA in D and E respectively, and the angle between EC and CD is equal to the angle C (Chap. I., Art. 6).

Now, from the two triangles ECD and EOD we have

$$DE^2 = CD^2 + CE^2 - 2\,CD\,.\,CE\,.\cos C \;.\;.\;. \qquad (\alpha)$$

Also $$DE^2 = OD^2 + OE^2 - 2\,OD\,.\,OE\,.\cos c\;.\;.\;.; \qquad (\beta)$$

and since the angles OCE and OCD are each 90°,

$$OC^2 = OD^2 - CD^2 = OE^2 - CE^2.$$

Hence, on subtracting (α) from (β),

$$0 = OC^2 + CD\,.\,CE \cos C - OD\,.\,OE\,.\cos c;$$

or,

$$\cos c = \frac{OC}{OD}\cdot\frac{OC}{OE} + \frac{CD}{OD}\cdot\frac{CE}{OE}\cdot\cos C;$$

$$\left.\begin{aligned} &\text{therefore,} && \cos c = \cos a \cos b + \sin a \sin b \cos C. \\ &\text{Similarly,} && \cos b = \cos c \cos a + \sin c \sin a \cos B, \\ &\text{and} && \cos a = \cos b \cos c + \sin b \sin c \cos A. \end{aligned}\right\} \qquad (1)$$

From these formulæ we obtain the angles in terms of the sides. Thus:

$$\left.\begin{aligned} \cos A &= \frac{\cos a - \cos b \cos c}{\sin b \sin c} \;.\;.\;.; \\ \cos B &= \frac{\cos b - \cos c \cos a}{\sin c \sin a} \;.\;.\;.; \\ \cos C &= \frac{\cos c - \cos a \cos b}{\sin a \sin b} \;\;.\;.\;. \end{aligned}\right\} \qquad (2)$$

These results are of special importance, and the student should make himself perfectly familiar with them at the outset.

They apply to all spherical triangles, whether the arcs be greater or less than quadrants.

27. Generalization.—In the preceding Article it is plain, on referring to the figure, that the proof given only applies to a spherical triangle having the sides a and b each less than $\frac{1}{2}\pi$. For, in the two triangles OCE and OCD, each right-angled at C, the angles COE and COD are acute, and therefore the arcs a and b subtended by them on the sphere are less than $\frac{1}{2}\pi$. No such restriction has been placed on the limits of the remaining side c: it may, therefore, be greater, equal to, or less than, $\frac{1}{2}\pi$.

It therefore remains to be shown that the foregoing theorem is applicable when *either of* or *both* the sides a and b are greater than quadrants.

(1). *In the former case* let a be greater than, and b less than, a quadrant.

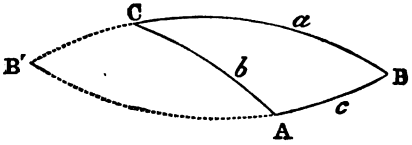

Fig. 14.

Let a and c (fig. 14) be produced to meet in B'. Then in the triangle $AB'C$, by the preceding it follows that

$$\cos AB' = \cos B'C \cos AC + \sin B'C \sin AC \cos ACB';$$

therefore,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

(2). *In the latter case* let a and b (fig. 15) be each greater than a quadrant

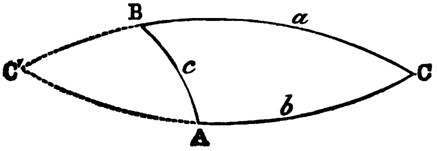

Fig. 15.

Then in the triangle ABC' we have

$$\cos c = \cos BC' \cos AC' + \sin BC' \sin AC' \cos C;$$

or,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

The remaining cases, when either of or both the sides a and b are quadrants, are left as easy exercises for the student. The formulæ referred to are, therefore, applicable to all triangles drawn on the surface of a sphere.

27 (*a*).—**Right-angled Triangle.**—We shall now apply the formula of Art. 26 to a triangle right-angled at C, and deduce the relations existing amongst the parts of such a triangle.

Since $C = \frac{1}{2}\pi$, $\cos C = 0$; hence

(1). $$\cos c = \cos a \cos b.$$

(2). Again, from the formula,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

On substituting the value of $\cos a$ from Ex. (1) we have

$$\cos A = \frac{\tan b}{\tan c}.$$

Similarly,

$$\cos B = \frac{\tan a}{\tan c}.$$

(3). On substituting the value of $\cos b$ from Ex. (1) we have, from (2), after reduction,

$$\sin A = \frac{\sin a}{\sin c}.$$

Similarly,

$$\sin B = \frac{\sin b}{\sin c}.$$

(4). From Examples (2) and (3) we have, by division,

$$\tan a = \tan A \sin b,$$

and

$$\tan b = \tan B \sin a.$$

(5). From (4) we have

$$\tan a \tan b = \tan A \tan B \;.\; \sin a \sin b;$$

or,

$$\cot A \cot B = \cos a \cos b = \cos c.$$

(6). From (2) and (3) we have

$$\cos A = \cos a \sin B.$$

The above six formulæ, connecting the sides and angles of a right-angled triangle, are expressed in general terms thus:—

(1) cos (hypotenuse) = product of cosines of sides.
(2) cos (angle) = tangent (adjacent side) ÷ tan (hypotenuse).
(3) sin (angle) = sin (opposite side) ÷ sin (hypotenuse).
(4) tan (side) = tan (opposite angle) × sin (remaining side).
(5) cos (hypotenuse) = product of cotangents of base angles.
(6) cos (angle) = cos (opposite side) × sin (remaining angle).

(7). Let $b = c = \frac{\pi}{2}$, and the expression

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

reduces to

$$\cos a = \cos A, \qquad [\textit{Vide} \text{ Art. 6, fig. 2.}]$$

as is otherwise evident, since in this case C is the pole of AB.

(8). Let $a = b = c$;

then $\cos a = \cos^2 a + \sin^2 a \cos A$;

or, $\cos a (1 - \cos a) = (1 - \cos^2 a) \cos A.$

Hence, $\sec A = 1 + \sec a,$

a constant relation between the angle and side of an equilateral spherical triangle, which shows that when the sides are quadrants the angles are right angles, and conversely.

(9). Given a side or angle of an equilateral triangle; solve it completely, and deduce the limiting values of the sides and angles of an equilateral triangle. [Apply Ex. (8).]

(10). Given the latitudes and difference of longitudes of two places on the Earth's surface, compute the distance between them.

—(*Science and Art Exam. Papers*).

(11). Apply Art. 26 (1) to prove the theorem of Art. 19 (3).

28. **Analogous Formula in Plane.**—Supposing the radius of the sphere to be *indefinitely great*, show that the formula

$$\frac{\cos a - \cos b \cos c}{\sin b \sin c} = \cos A$$

degenerates into

$$\frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} = \cos A,$$

where A is an angle, and α, β, and γ the sides of a plane triangle.

Expanding $\cos a$, &c., by the aid of formulæ, Chap. II., Art. 18, and neglecting all powers of the radius above the second, we have

$$\frac{\left(1 - \frac{\alpha^2}{\underline{|2}\,r^2}\right) - \left(1 - \frac{\beta^2}{\underline{|2}\,r^2}\right)\left(1 - \frac{\gamma^2}{\underline{|2}\,r^2}\right)}{\frac{\beta}{r} \cdot \frac{\gamma}{r}} = \cos A,$$

which reduces to

$$\frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} = \cos A.$$

29. **Legendre's Theorem.**—*If the sides of a spherical triangle be small compared with the radius of the sphere, then each angle of the spherical triangle exceeds by one-third of the spherical excess the corresponding angle of the plane triangle, the sides of which are of the same length as the arcs of the spherical triangle.*

In the foregoing example the powers of the radius higher than the *second* were neglected. Suppose, however, a closer approximation to be made, and the powers of $\frac{1}{r}$ above the *fourth* neglected.

Then

$$\cos A = \frac{1 - \frac{\alpha^2}{2r^2} + \frac{\alpha^4}{24r^4} - \left(1 - \frac{\beta^2}{2r^2} + \frac{\beta^4}{24r^4}\right)\left(1 - \frac{\gamma^2}{2r^2} + \frac{\gamma^4}{24r^4}\right)}{\frac{\beta}{r}\left(1 - \frac{\beta^2}{6r^2}\right)\frac{\gamma}{r}\left(1 - \frac{\gamma^2}{6r^2}\right)}$$

$$= \frac{\frac{\beta^2 + \gamma^2 - \alpha^2}{2r^2} + \frac{\alpha^4 - \beta^4 - \gamma^4 - 6\beta^2\gamma^2}{24r^4}}{\frac{\beta\gamma}{r^2}\left(1 - \frac{\beta^2 + \gamma^2}{6r^2}\right)}.$$

But by the aid of the Binomial Theorem we have

$$\frac{1}{1-\frac{\beta^2+\gamma^2}{6r}}=\left(1-\frac{\beta^2+\gamma^2}{6r^2}\right)^{-1}=1+\frac{\beta^2+\gamma^2}{6r^2},\text{ nearly.}$$

Hence

$$\cos A=\frac{\left\{\frac{\beta^2+\gamma^2-\alpha^2}{2r^2}+\frac{\alpha^4-\beta^4-\gamma^4-6\beta^2\gamma^2}{24r^4}\right\}\left\{1+\frac{\beta^2+\gamma^2}{6r^2}\right\}}{\frac{\beta\gamma}{r^2}}$$

$$=\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}-\frac{2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4}{24\beta\gamma r^2}.$$

Now consider a plane triangle of sides, α, β, and γ; angles A', B', and C'; and area $= \Delta = \frac{1}{2}\beta\gamma \sin A'$.

The above equation reduces to the form (Plane Trig.)

$$\cos A=\cos A'-\frac{\sin^2 A'\,.\,\beta\gamma}{6r^2}.$$

Let the excess of A over A' be denoted by θ, θ being a very small quantity; then $A = A' + \theta$, and

$$\cos A=\cos A'-\theta\sin A'\text{ nearly;}$$

therefore,

$$\cos A'-\frac{\sin^2 A'}{6r^2}\beta\gamma=\cos A'-\theta\sin A';$$

or,

$$\theta=\frac{\Delta}{3r^2}.$$

We have, therefore,

$$A=A'+\frac{\Delta}{3r^2};\qquad B=B'+\frac{\Delta}{3r^2};\qquad C=C'+\frac{\Delta}{3r^2}.$$

Whence

$$A+B+C-A'-B'-C'=\frac{\Delta}{r^2},$$

or,

$$A+B+C-\pi=\frac{\Delta}{r^2}.$$

But the expression $A+B+C-\pi$ denotes the excess of the sum of the angles of a spherical triangle over the sum of the angles of a plane triangle (both being expressed in circular measure), and is therefore called the *Spherical Excess*; therefore, &c.

[*See* also *Exam. Paper* VIII. 2, 3; and Art. 107.]

SECTION II.

30. *Expression for the sine of an angle of a spherical triangle in terms of the trigonometrical functions of the sides*

We have (Art. 26),

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

therefore

$$\sin^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$

$$= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}$$

$$= \frac{4n^2}{\sin^2 b \sin^2 c},$$

where

$$4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c.\text{*}$$

$$\left.\begin{array}{ll} \text{Hence,} & \sin A = \dfrac{2n}{\sin b \sin c}. \\ \text{Similarly,} & \sin B = \dfrac{2n}{\sin c \sin a}, \\ \text{and} & \sin C = \dfrac{2n}{\sin a \sin b}. \end{array}\right\} \qquad (3)$$

---

* The function n has been called the sine of the solid angle that the triangle subtends at the centre of the sphere (*vide* Salmon's *Geometry of Three Dimensions*, Art. 54), and may be written in the determinant form

$$4n^2 = \begin{vmatrix} 1, & \cos c, & \cos b \\ \cos c, & 1, & \cos a \\ \cos b, & \cos a, & 1, \end{vmatrix}.$$

31. Rule of Sines.—From the value of sin A in the previous Artice we have at once

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2n}{\sin a \sin b \sin c}. \qquad (4)$$

*Hence,** *the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.*

The corresponding theorem *in plano* is, "The Sines of the Angles of a Plane Triangle are proportional to the opposite Sides."

Examples.

1. Given the base c and the function n, find the locus of the vertex.
—(*Q. U. I., Exam. Papers.*)

Since $$\sin A = \frac{2n}{\sin b \sin c},$$

we have given $\sin b \sin A$, which, by the aid of Art. 31, or 27, Ex. 3, is equal to $\sin p$, where p is the perpendicular from the vertex on the base. The locus is, therefore, a small circle having the same poles as the base.

2. If α, β, γ, be the perpendiculars of a triangle, prove that

$$\sin a \sin \alpha = \sin b \sin \beta = \sin c \sin \gamma = 2n.$$

[For (Art. 27, Ex. 3),

$$\sin \alpha = \sin b \sin C, \quad \sin \beta = \sin c \sin A, \quad \sin \gamma = \sin a \sin B;$$

therefore, &c., Art. 30 (3).]

Note.—The rule of sines follows at once from the equation

$$\sin \alpha = \sin b \sin C = \sin c \sin B;$$

therefore

$$\frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}.$$

Therefore, &c.

---

* Or thus:—take P, the pole of the side c, and apply equation (1), Art. 26, to the triangles ACP and BCP. Then $\cos CP = \sin a \sin B = \sin b \sin A$. A particular case of this theorem has already been proved in Art. 24.

3. In Ex. 2 show that

$$\sin a \cos \alpha = \sqrt{\cos^2 b + \cos^2 c - 2 \cos a \cos b \cos c},$$

with similar expressions for $\cos \beta$ and $\cos \gamma$.

[Since

$$\sin \alpha = \sin b \sin C = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a}.$$

Hence $\sin^2 a \sin^2 \alpha$, or $\sin^2 a - \sin^2 a \cos^2 \alpha$

$$= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c;$$

therefore, &c.]

4. The sines of the segments of the base of a triangle made by the bisectors of the internal and external vertical angle are to one another as the sines of the adjacent sides. (Cf. Euc., VI. III.)

5. The bisector of the base divides the vertical angle into two parts, the ratio of whose sines is the inverse ratio of the sines of the adjacent sides.

6. If M be the middle point of the base AB of a triangle ABC, and M' * a point on the base, such that the angle ACM is equal to the angle BCM', prove that

$$\sin AM' : \sin BM' :: \sin^2 b : \sin^2 a.$$

7. If p_1, p_2 be the perpendiculars drawn from the mid-point M of the base of a spherical triangle on the great circle bisectors of the vertical angle A, and p_3 the perpendicular from A on the great circle perpendicular to the base through M, prove—

(1) $$\sin p_1 \sin p_2 = \tfrac{1}{2} \sin p_3 \sin \frac{a}{2} \sin (B + C).$$

(2) Give the analogous theorem for a plane triangle.

—(*Educational Times*, August, 1884.)

[Let x and y be the points where the external and internal bisectors meet the perpendicular through M; then

$$\frac{\sin p_1 \sin p_2}{\sin p_3} = \frac{\sin Mx \sin x \,.\, \sin My \sin y}{\sin Ax \,.\, \sin x} = \frac{\sin Mx \,.\, \sin My}{\sin xy};$$

therefore, &c.

The analogous theorem *in plano* is $\dfrac{4p_1p_2}{p_3} = a \sin A$.]

---

* The arc CM' is called a *Symmedian* of the triangle ABC.

32. Geometrical Proof of the Rule of Sines.—*To show geometrically that the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides.*

Let ABC (fig. 16) be a spherical triangle, O the centre of the sphere. From any point P in OC let fall a perpendicular PD on the plane AOB. Join OD, and let fall DE perpendicular on OA. Join PE. PD is perpendicular to all lines drawn through D in the plane AOB.

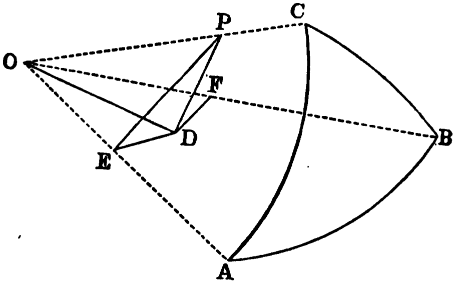

Fig. 16.

Now

$$OP^2 = OD^2 + PD^2 = OE^2 + DE^2 + PD^2 = OE^2 + PE^2;$$

therefore PEO is a right angle.

Now, since PE and DE are drawn in the planes OAC and OAB perpendicular to their line of intersection OA,

$$\angle PED = \angle A;$$

therefore

$$\sin A = \frac{PD}{PE}, \quad \sin b = \frac{PE}{OP},$$

and

$$\sin A \sin b = \frac{PD}{OP}.$$

Similarly, by letting fall a perpendicular DF on OB we get

$$\sin B \sin a = \frac{PD}{OP} = \sin A \sin b;$$

hence

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

33. Expressions for the sine, cosine, and tangent of half an angle of a triangle as functions of the sides.

We have

$$2\cos^2\frac{A}{2} = 1 + \cos A$$

$$= 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c} \qquad \text{[Art. 26 (2)].}$$

$$= \frac{\cos a - \cos (b+c)}{\sin b \sin c} = \frac{2 \sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(b+c-a)}{\sin b \sin c}$$

$$= \frac{2 \sin s \sin (s-a)}{\sin b \sin c}, \quad \text{where } \underline{2s = a+b+c;}$$

hence

$$\left.\begin{array}{ll} & \cos\dfrac{A}{2} = \sqrt{\dfrac{\sin s \sin (s-a)}{\sin b \sin c}}. \\ \text{Similarly,} & \\ & \cos\dfrac{B}{2} = \sqrt{\dfrac{\sin s \sin (s-b)}{\sin c \sin a}}, \\ \text{and} & \\ & \cos\dfrac{C}{2} = \sqrt{\dfrac{\sin s \sin (s-c)}{\sin a \sin b}}. \end{array}\right\} \qquad (5)$$

Again,

$$2\sin^2\frac{A}{2} = 1 - \cos A$$

$$= 1 - \frac{\cos a - \cos b\cos c}{\sin b\sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b\sin c}$$

$$= \frac{2\sin(s-b)\sin(s-c)}{\sin b\sin c};$$

hence

$$\sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.$$

Similarly,

$$\sin\frac{B}{2} = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c\sin a}},\quad (6)$$

and

$$\sin\frac{C}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}.$$

From the above results we obtain

$$\tan\frac{A}{2} = \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}},$$

with similar expressions for $\tan\frac{B}{2}$ and $\tan\frac{C}{2}$. (7)

Note.—

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2}{\sin b\sin c}\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)};$$

[From (5) and (6)].

hence

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}=\frac{2}{\sin a\sin b\sin c}\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}.$$

Comparing this expression for $\sin A$ with that derived in Art. 30, we see that

$$\begin{aligned}&\sin s.\sin(s-a).\sin(s-b).\sin(s-c)\\&=\tfrac{1}{4}(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c)\\&=n^2,\end{aligned}$$

an *identity* which can be readily proved by direct multiplication of the four factors, $\sin s$, $\sin(s-a)$, $\sin(s-b)$, and $\sin(s-c)$.

Thus,

$$2\sin s\sin(s-a)=\cos a-\cos(b+c),\text{ and } 2\sin(s-b)\sin(s-c)=\cos(b-c)-\cos a;$$

therefore, &c.

Examples.

1. Find the conditions that

$$1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c=0.$$

2. Prove that

$$8n^3=\sin^2 a\sin^2 b\sin^2 c\sin A\sin B\sin C.$$

3. Prove that

$$\cot\frac{A}{2}:\cot\frac{B}{2}:\cot\frac{C}{2}=\sin(s-a):\sin(s-b):\sin(s-c).$$

4. Prove that

$$\tan\frac{B}{2}\tan\frac{C}{2}=\frac{\sin(s-a)}{\sin s}.$$

5. Prove that

$$\sin(s-a)=\frac{\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A}\sin a,$$

with similar expressions for $\sin(s-b)$ and $\sin(s-c)$.

6. Prove that

$$\sin s = \frac{\cos \frac{1}{2}B \cos \frac{1}{2}C}{\sin \frac{1}{2}A} \sin a.$$

7. Prove that

$$\sin a \sin b \sin c \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{n^2}{\sin s}.$$

8. Prove that

$$\sin \frac{A}{2} \cos \frac{B}{2} = \frac{\sin (s-b)}{\sin c} \cos \frac{C}{2}.$$

Note.[*]—By the aid of a similar equation,

$$\cos \frac{A}{2} \sin \frac{B}{2} = \frac{\sin (s-a)}{\sin c} \cos \frac{C}{2}.$$

We have, on adding and subtracting,

$$\sin \frac{A+B}{2} \cos \frac{c}{2} = \cos \frac{a-b}{2} \cos \frac{C}{2}. \qquad (\alpha)$$

$$\sin \frac{A-B}{2} \sin \frac{c}{2} = \sin \frac{a-b}{2} \cos \frac{C}{2}. \qquad (\beta)$$

Again, from Examples 5 and 6, we have

$$\cos \frac{A+B}{2} \cos \frac{c}{2} = \cos \frac{a+b}{2} \sin \frac{C}{2}. \qquad (\gamma)$$

$$\cos \frac{A-B}{2} \sin \frac{c}{2} = \sin \frac{a+b}{2} \sin \frac{C}{2}, \qquad (\delta)$$

four equations which will be afterwards proved geometrically.

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[*] The equations α, β, γ, δ, are known as Gauss's Formulæ, and the equations α', β', γ', δ' (Ex. 9), are known as Napier's Analogies. They will all be further discussed later on.

9. Prove that—

(α′). $$\tan\frac{A+B}{2} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

(β′). $$\tan\frac{A-B}{2} = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

(γ′). $$\tan\frac{a+b}{2} = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

(δ′). $$\tan\frac{a-b}{2} = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

[These results follow from Ex. 8.]

SUPPLEMENTAL THEOREMS.

34. *Having given the angles of a spherical triangle, to determine the cosines of the sides* (cf. Art. 26).

In the formula,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

change the sides into the supplements of the corresponding angles, and the angle into the supplement of the corresponding side (Note, Art. 22), and we get

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a.$$

Therefore,

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

Similarly,

$$\cos b = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

and

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

(8)

All the results obtained in Art. 26, as particular cases of formula (1) of this Chapter, may also be deduced as particular cases of formula (8), by supposing the triangle to be equilateral or right-angled, as the case may require.

35. **Analogous Formula in Plano.**—If we suppose the radius of the sphere to be indefinitely great, we have (Chap. II., Art. 18) $\cos a = 1$. Therefore,

$$\cos A + \cos B \cos C = \sin B \sin C,$$

from which it follows at once that

$$A + B + C = \pi.$$

Examples.

1. In any triangle show that

$$\frac{\cos A + \cos B}{1 - \cos C} = \frac{\sin (a+b)}{\sin c}.$$

[If the internal bisector of the angle C make an angle θ with the opposit side, we have

$$\cos \theta + \cos A \cos \frac{C}{2} = \sin A \sin \frac{C}{2} \cos b; \text{ by Art. 34 (8);}$$

and also

$$-\cos \theta + \cos B \cos \frac{C}{2} = \sin B \sin \frac{C}{2} \cos a.$$

Adding these equations, we eliminate θ, and deduce the above expression.]

2. In any triangle show that

$$\Sigma \frac{\cos A + \cos B}{1 - \cos C} \sin (a - b) \sin c = 0.$$

[This result may be obtained at once from Ex. 1.]

3. Prove the relation

$$\frac{\cos A - \cos B}{1 + \cos C} = \frac{\sin (a \sim b)}{\sin c}.$$

[This follows, as in Ex. 1, by drawing the external, instead of the internal, bisector of C.]

4. Prove the relation

$$\Sigma \frac{\cos A - \cos B}{1 + \cos C} \sin (a + b) \sin c = 0.$$

5. If a great circle passes through the vertex C, making angles α and β with the sides a and b, the angle θ, which it makes with the side c, is given by the equation

$$\cos A \sin \alpha - \cos B \sin \beta = \pm \cos \theta \sin C.$$

[For

$$\cos A + \cos \theta \cos \beta = \sin \theta \sin \beta \cos \delta \text{ (Art. 34 (8))},$$

where δ is the intercept between the vertex and the base; also

$$\cos B - \cos \theta \cos \alpha = \sin \theta \sin \alpha \cos \delta,$$

eliminate δ between these equations; therefore, &c.]

6. If through any point P on a sphere three great circles be drawn, cutting the sides of a triangle at angles X, Y, Z; X_1, Y_1, Z_1; X_2, Y_2, Z_2, respectively; prove the following *determinant** relation :—

$$\begin{vmatrix} \cos X & \cos Y & \cos Z \\ \cos X_1 & \cos Y_1 & \cos Z_1 \\ \cos X_2 & \cos Y_2 & \cos Z_2 \end{vmatrix} = 0.$$

[Let the three concurrent arcs make angles α, β, and $\alpha + \beta$ with each other.

Since the side a is cut at angles X, X_1, and X_2; by Ex. 5,

$$\cos X \sin \alpha - \cos X_2 \sin \beta = \cos X_1 \sin (\alpha + \beta).$$

Similarly,

$$\cos Y \sin \alpha - \cos Y_2 \sin \beta = \cos Y_1 \sin (\alpha + \beta),$$

and

$$\cos Z \sin \alpha - \cos Z_2 \sin \beta = \cos Z_1 \sin (\alpha + \beta).$$

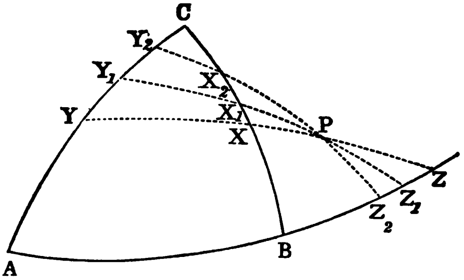

Fig. 17.

Eliminating α and β from the three equations, the above result easily follows.]

---

* For a knowledge of Determinants, *vide* Burnside and Panton's *Theory of Equations*, Chap. xi.

7. Having given that the sides of a triangle are each $\frac{\pi}{3}$, find the sides of the supplemental triangle.

$\left[\text{Since the angles of the latter triangle are each } \frac{2\pi}{3},\ \cos a = -\frac{1}{3}\right].$

8. Find a relation connecting the angles of a triangle if one side a is a quadrant. *Ans.* $\cos A + \cos B \cos C = 0$.

9. Given A, B, C, find the angle θ which the bisector of the vertical angle makes with the base.

[If i be the length of the internal bisector, we have

$$\cos i = \frac{\cos A + \cos\theta \cos\frac{1}{2}C}{\sin\theta \sin\frac{1}{2}C} = \frac{\cos B - \cos\theta\cos\frac{1}{2}C}{\sin\theta\sin\frac{1}{2}C};$$

therefore,

$$\cos\theta = \frac{\cos A \sim \cos B}{2\cos\frac{1}{2}C}.\Big]$$

10. If θ' denote the angle made by the external bisector of the vertical angle with the base, show that

$$\cos\theta' = \frac{\cos A + \cos B}{2\sin\frac{1}{2}C}. \qquad \text{(Cf. Ex. 9.)}$$

11. From any three points, A, B, and C, on a great circle, secondaries AA', BB', and CC' are drawn to another great circle; prove that the *algebraic* sum

$$\sin B'C' \cos A + \sin C'A' \cos B + \sin A'B' \cos C$$

is equal to zero. (Apply Ex. 5.)

12. The extremities of the diameter AB of a small circle are joined with a point C on the circle; prove that the angles subtended at the pole by the joining arcs AC and BC are $\cos^{-1}\frac{\sin(A \sim B)}{\sin C}$. (Apply Ex. 5.)

13. Given the base c and $\frac{\cos A}{\cos B} = -\cos C$, find the locus of the vertex.

[$a = \frac{\pi}{2}$ by Ex. 8. Hence the locus is a great circle, having the vertex B for pole.]

36. Expression for the Side of a Spherical Triangle in terms of the Trigonometrical Functions of the Angles. (Cf. Art. 33.)

We have

$$2\sin^2\frac{a}{2} = 1 - \cos a = 1 - \frac{\cos A + \cos B \cos C}{\sin B \sin C} \qquad \text{[Art. 34.]}$$

$$= -\frac{\cos A + \cos(B + C)}{\sin B \sin C} = -\frac{2\cos S . \cos(S - A)}{\sin B \sin C},$$

where

$$2S = A + B + C.$$

Therefore,

$$\sin\frac{a}{2} = \sqrt{\frac{-\cos S \cos(S - A)}{\sin B \sin C}}, \qquad (9)$$

with similar values for sin $\frac{1}{2}b$ and sin $\frac{1}{2}c$

Again,

$$2\cos^2\frac{a}{2} = 1 + \cos a = 1 + \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

$$= \frac{\cos A + \cos(B - C)}{\sin B \sin C} = \frac{2\cos(S - B)\cos(S - C)}{\sin B \sin C};$$

therefore,

$$\cos\frac{a}{2} = \sqrt{\frac{\cos(S - B)\cos(S - C)}{\sin B \sin C}}, \qquad (10)$$

with similar values for cos $\frac{1}{2}b$ and cos $\frac{1}{2}c$.

Also

$$\tan\frac{a}{2} = \frac{\sin\frac{1}{2}a}{\cos\frac{1}{2}a} = \sqrt{\frac{-\cos S \cos(S - A)}{\cos(S - B)\cos(S - C)}}, \qquad (11)$$

with similar values for tan $\frac{1}{2}b$ and tan $\frac{1}{2}c$.

Moreover,

$$\sin a = 2 \sin \tfrac{1}{2} a \cos \tfrac{1}{2} a$$

$$= \frac{2}{\sin B \sin C} \sqrt{-\cos S \cos (S - A) \cos (S - B) \cos (S - C)}. \qquad (12)$$

We shall use the symbol N to denote the radical in this expression, so that we have

$$\sin B \sin C \sin a = \sin C \sin A \sin b = \sin A \sin B \sin c = 2N.$$

Remark.—The positive sign has been given to the radicals in the formulæ for $\sin \frac{1}{2}a$, $\cos \frac{1}{2}a$, and $\tan \frac{1}{2}a$, since $\frac{1}{2}a$ is necessarily less than $\frac{1}{2}\pi$. (See also Chap. II., Art. 23.)

Examples.

1. $$4N^2 = 1 - \cos^2 A - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C.$$

2. If p, q, r be the perpendiculars from the vertices on the opposite sides, show that—

(α). $\sin a \; \sin p = \sin b \; \sin q = \sin c \; \sin r = 2n.$

(β). $\sin A \sin p = \sin B \sin q = \sin C \sin r = 2N$;

and hence

$$\frac{\sin a}{\sin A} = \text{etc.} = \frac{n}{N}$$

3. Prove that

$$n = \frac{2N^2}{\sin A \sin B \sin C}.$$

4. Prove that

$$N = \frac{2n^2}{\sin a \sin b \sin c}.$$

5. Prove that

$$2N = (\sin a \sin b \sin c \sin^2 A \sin^2 B \sin^2 C)^{\frac{1}{3}}.$$

6. Prove that

$$\tan\frac{b}{2}\tan\frac{c}{2}=\frac{-\cos S}{\cos(S-A)}.$$

[Geometrically thus (see Chap. II., fig. 10):

$$\frac{-\cos S}{\cos(S-A)}=\frac{\tan AL}{\tan AD}:\frac{\tan AL}{\tan AY}=\cot AD\tan AY=\tan\frac{b}{2}\tan\frac{c}{2}.$$

7. Prove that

$$\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\sin^2 a+\sin^2 b+\sin^2 c}=\frac{1+\cos A\cos B\cos C}{1-\cos a\cos b\cos c}.$$

—(*Dublin Univ. Exam. Papers.*)

[We have

$$\frac{N^2}{n^2}=\frac{\sin^2 A}{\sin^2 a}=\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\sin^2 a+\sin^2 b+\sin^2 c}=\frac{4N^2-\sin^2 A-\sin^2 B-\sin^2 C}{4n^2-\sin^2 a-\sin^2 b-\sin^2 c};$$

therefore, &c. (see Exs. 1 and 2).]

8. Given the vertical angle C of a triangle, and the function N, find the *envelope** of the base.

[By formula (12)

$$2N=\sin B\sin C\sin a=\sin C\sin r,$$

where r is the perpendicular from the vertex on the base. The envelope is, therefore, a small circle round C as pole.]

9. If $A=a$, then B and C are respectively equal or supplemental to b and c, and $N=n$.

10. Show that

$$4Nn=\sin a\sin b\sin c\sin A\sin B\sin C.$$

11. Verify the following formulæ by the expansions of $\tan\frac{1}{2}(a\pm b)$:—

$$\alpha^\circ.\quad \tan\frac{a+b}{2}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

$$\beta^\circ.\quad \tan\frac{a-b}{2}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

---

* The envelope is the figure which it always touches.

12. Write out and otherwise establish the corresponding formulæ to (α° and β°, Ex. 11) for the supplemental triangle.

13. If l be the length of the arc joining the middle point of the base to the vertex, find an expression for its length in terms of the sides.

[Let the arc l make an angle θ with the base c. Then, Art. 26,

$$\cos\theta = \frac{\cos a - \cos l \cos\frac{1}{2}c}{\sin l \sin\frac{1}{2}c} = -\frac{\cos b - \cos l \cos\frac{1}{2}c}{\sin l \sin\frac{1}{2}c};$$

therefore,

$$\cos l = \frac{\cos a + \cos b}{2\cos\frac{1}{2}c}. \Big]$$

14. What is the corresponding theorem *in plano* to Ex. 13?

[Let α, β, γ, and λ be the *lengths* of the arcs a, b, c, and l, respectively. Expanding by Chap. II., Art. 18, and neglecting the powers of the radius higher than the second, we have

$$1 - \frac{\lambda^2}{2r^2} = \frac{\left(1 - \frac{\alpha^2}{2r^2}\right)\left(1 - \frac{\beta^2}{2r^2}\right)}{2\left(1 - \frac{\gamma^2}{8r^2}\right)}.$$

Hence, by reduction,

$$\alpha^2 + \beta^2 = 2\left(\frac{\gamma}{2}\right)^2 + 2\lambda^2,$$

a result well known in Geometry.]

15. Having given the base, and the sum of the cosines of the sides, find the locus of the vertex. (Cf. Ex. 13.)

SECTION III.

Three Important Theorems.

37. **Theorem I.**—In any spherical triangle,

$$\cot a \sin b = \cot A \sin C + \cos b \cos C.\text{*}$$

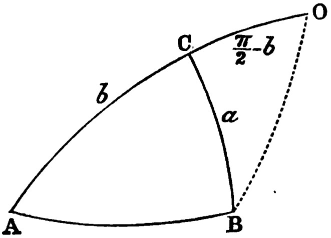

Fig. 18.

Let ABC (fig. 18) be a spherical triangle. Produce AC through C to O, so that $AO = \frac{\pi}{2}$. Join BO.

Applying Art. 26 (1), to the triangles BOC and BOA, we have

$$\cos BO = \cos a \sin b - \sin a \cos b \cos C$$

from triangle BOC; and

$$\cos BO = \sin c \cos A$$

from triangle BOA, since $\cos AO = 0$.

---

[*] The following symmetrical form of writing this equation, in which it is easily remembered, has been afloat for some time, and is worthy of notice. Taking the four parts of the triangle connected by the equation in the order in which they occur in the triangle (e.g. a, C, b, A), and calling the first and fourth the extreme, and the second and third the middle terms, the formula becomes

$$\begin{aligned}\cos(\text{mid. side})\cos(\text{mid. angle}) &= \sin(\text{mid. side})\cot(\text{other side})\\ &\quad - \sin(\text{mid. angle})\cot(\text{other angle}).\end{aligned}$$

Hence, equating these expressions for $\cos BO$,

$$\cos a \sin b = \sin a \cos b \cos C + \sin c \cos A.$$

Dividing each side of this equation by $\sin a$, and substituting $\frac{\sin C}{\sin A}$ for $\frac{\sin c}{\sin a}$, and we have

$$\cot a \sin b = \cot A \sin C + \cos b \cos C. \qquad (1)$$

Similarly, by producing the side BC through C to a point $\frac{\pi}{2}$ from B, we have

$$\cot b \sin a = \cot B \sin C + \cos a \cos C. \qquad (2)$$

Similarly, by producing the sides of the triangle through the angles A and B, it follows that

$$\cot b \sin c = \cot B \sin A + \cos c \cos A. \qquad (3)$$

$$\cot c \sin b = \cot C \sin A + \cos b \cos A. \qquad (4)$$

$$\cot c \sin a = \cot C \sin B + \cos a \cos B. \qquad (5)$$

$$\cot a \sin c = \cot A \sin B + \cos c \cos B. \qquad (6)$$

The same results will be obtained directly from the equations

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

. . . .

by eliminating the sides a or c.

The above formulæ, as they enable us, when given two sides and an angle, or two angles and a side of a triangle, to determine the remaining parts, are of great importance, and are thus expressed in general terms:—

Cotangent of any side × sine of another side = cotangent of the angle opposite to the first side × sine of included angle + cosine of second side × cosine of included angle.

38. Applying equation (1) of preceding Article to the polar triangle, we have

$$-\cot A \sin B = -\cot a \sin c + \cos B \cos c;$$

or, by transposing,

$$\cot a \sin c = \cot A \sin B + \cos c \cos B,$$

an equation identical with (6), Art. 37.

Moreover, it will be seen that each of the formulæ just proved contains four parts of a spherical triangle, which are connected thus:—

(*a*) Two sides and included angle with one of the base angles; or,

(β) Two angles and an adjacent side with one of the remaining sides.

39. **Corresponding Theorem in Plane.**—Using equation (1), and putting

$$\frac{r}{a} = \cot a, \quad \frac{\beta}{r} = \sin b, \quad \text{and } \cos b = 1,$$

we have

$$\frac{\beta}{a} = \cot A \sin C + \cos C = \frac{\sin(A + C)}{\sin A}.$$

Hence

$$\frac{a}{\beta} = \frac{\sin A}{\sin B}.$$

(*Science and Art Math. Honors Exam.*)

Examples.

1. **Having given two sides of a spherical triangle, find the length of the bisector of the vertical angle intercepted between the vertex and the base.**

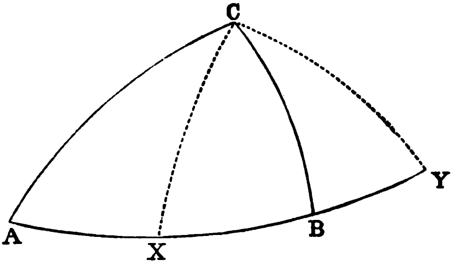

Fig. 19.

Let ι denote the internal bisector of the angle C, and θ (fig. 19) the angle it makes with the base.

Then

$$\cot a \sin \iota = \cot \theta \sin \tfrac{1}{2} C + \cos \iota \cos \tfrac{1}{2} C,$$

and

$$\cot b \sin \iota = - \cot \theta \sin \tfrac{1}{2} C + \cos \iota \cos \tfrac{1}{2} C;$$

by addition,

$$(\cot a + \cot b) \sin \iota = 2 \cos \iota \cos \tfrac{1}{2} C;$$

or

$$\cot \iota = \frac{\cot a + \cot b}{2 \cos \frac{1}{2} C}.$$

Similarly, if η denote the external bisector of C,

$$\cot \eta = \frac{\cot a \sim \cot b}{2 \sin \frac{1}{2} C}.$$

2. **What are the analogous formulæ *in plano*?**

Ans. $\dfrac{1}{\text{Internal Bisector}} = \dfrac{\frac{1}{a} + \frac{1}{b}}{2 \cos \frac{1}{2} C}$, or Internal Bisector $= \dfrac{2ab}{a+b} \cos \dfrac{C}{2}$.

and External Bisector $= \dfrac{2ab}{a \sim b} \sin \dfrac{C}{2}$.

3. If ι_1, ι_2, and ι_3 denote the bisectors of the internal angles of a triangle, prove that

$$\cot\iota_1\cos\frac{A}{2}+\cot\iota_2\cos\frac{B}{2}+\cot\iota_3\cos\frac{C}{2}=\cot a+\cot b+\cot c;$$

and hence, in a plane triangle (*vide* Hudson's *Plane Trig.*, p. 154),

$$\frac{\cos\frac{1}{2}A}{\iota_1}+\frac{\cos\frac{1}{2}B}{\iota_2}+\frac{\cos\frac{1}{2}C}{\iota_3}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c};$$

and if η_1, η_2, and η_3 denote the bisectors of the external angles,

$$\cot\eta_1\sin\frac{A}{2}+\cot\eta_2\sin\frac{B}{2}+\cot\eta_3\sin\frac{C}{2}=0.$$

4. Using the preceding notation, prove that

$$\cot\eta_1\sin\frac{A}{2}=\cot\iota_2\cos\frac{B}{2}\sim\cot\iota_3\cos\frac{C}{2}.$$

5. Find the angles θ and θ' (fig. 19) made by the bisectors of the angle C with the opposite side c.

[From the equations of Example 1 we have at once

$$\cot\theta=\frac{(\cot a-\cot b)}{2\sin\frac{1}{2}C}\sin\iota,$$

and

$$\cot\theta'=\frac{(\cot a+\cot b)}{2\cos\frac{1}{2}C}\sin\eta.]$$

6. Find the arc intercepted on the base by the bisectors of the vertical angle

$$\Big[\cos XY=\cot\theta\cot\theta' \quad \text{(fig. 19).}$$

$$=\frac{(\cot^2 a-\cot^2 b)}{2\sin C}\sin\iota\sin\eta; \quad \text{(Ex. 5)}$$

but $\sin\iota\sin\eta=\sin XY\sin r$ (Art. 31, Ex. 2), where r is the perpendicular from the vertex C on the base AB.

Therefore

$$\cot XY=\frac{(\cot^2 a-\cot^2 b)\sin a\sin b}{2\sin c} \quad \text{(Art. 31, Exs. 1 and 2.)}$$

$$=\frac{\sin(a+b)\sin(a-b)}{2\sin a\sin b\sin c};$$

or, in terms of the angles,

$$\cot XY=\frac{\sin^2 A-\sin^2 B}{2\sin A\sin B\sin c}.\Big]$$

7. Show that the sum of the cotangents of the intercepts made by the internal and external bisectors of the angles of a spherical triangle on the opposite sides is equal to zero. (*Educational Times.*)

[By the preceding Example

$$\Sigma \cot XY = \Sigma \frac{\sin^2 a - \sin^2 b}{2 \sin a \sin b \sin c} = 0.\Big]$$

8. Give the analogous property *in plano*.
(See Casey's *Sequel to Euclid*, sec. viii., Ex. 17.)

9. If ι_1, ι_2, and ι_3, the internal bisectors of the angles of a triangle, make angles respectively θ_1, θ_2, θ_3, with the opposite sides, prove that

$$\frac{\sin \frac{1}{2} A}{\sin \iota_1 \tan \theta_1} + \frac{\sin \frac{1}{2} B}{\sin \iota_2 \tan \theta_2} + \frac{\sin \frac{1}{2} C}{\sin \iota_3 \tan \theta_3} = 0;$$

(See Ex. 5.)

and that if θ_1', θ_2', and θ_3', are the angles made by the external bisectors with the opposite sides,

$$\frac{\cos \frac{1}{2} A}{\sin \eta_1 \tan \theta'_1} + \frac{\cos \frac{1}{2} B}{\sin \eta_2 \tan \theta'_2} + \frac{\cos \frac{1}{2} C}{\sin \eta_3 \tan \theta'_3} = \cot a + \cot b + \cot c.$$

(See Ex. 5.)

10. From the vertex C of a spherical triangle an arc of a great circle is drawn to meet the base AB in D; if C_1 and C_2 are the segments of the angle C, respectively, opposite AD and BD, show that

$$\cot CD \sin C = \cot a \sin C_1 + \cot b \sin C_2{}^*.$$

(*Science and Art Math. Honors.*)

[Let CD make an angle θ with the base, and we have

$$\cot a \sin CD = \cot \theta \sin C_2 + \cos CD \cos C_2 \qquad (1)$$

also,

$$\cot b \sin CD = - \cot \theta \sin C_1 + \cos CD \cos C_1 \qquad (2)$$

Eliminate θ from equations (1) and (2); therefore, &c.]

11. If ϕ be the angle between the base c of a spherical triangle and its bisector,

$$\cot A - \cot B = 2 \cos \frac{c}{2} \cot \phi.$$

---

[*] This relation connects the latitudes and *differences* of longitudes of three places in the same great circle on the surface of the Earth.

12. Prove that

$$\frac{\cos^2 b - \cos^2 c}{\cos b \cot B - \cos c \cot C} = \frac{\cos^2 c - \cos^2 a}{\cos c \cot C - \cos a \cot A}$$
$$= \frac{\cos^2 a - \cos^2 b}{\cos a \cot A - \cos b \cot B},$$

and express each of the three quantities as a symmetric function of the sides of the triangle.

[Multiplying equation (1) by $\cos a$ and (2) by $\cos b$, and subtracting the results, each of the expressions becomes $\sin b \sin c \sin A$, &c. $= 2n$.]

13. Write out the corresponding theorem for the supplemental triangle.

40. **Theorem II.**—*The angular distances of three points, A, B, and C, on the same great circle, and any other point, P, on the sphere, are connected by the relation*

$$\sin BC \cos AP + \sin CA \cos BP + \sin AB \cos CP = 0.$$

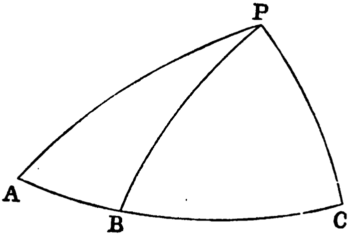

Fig. 20.

We have (fig. 20),*

$$\cos AP = \cos AB \cos BP + \sin AB \sin BP \cos ABP, \quad (1)$$

and

$$\cos CP = \cos BC \cos BP - \sin BC \sin BP \cos ABP. \quad (2)$$

---

[*] Or thus:—If D be the foot of the perpendicular p from P on AC, then $\Sigma \sin BC \cos AD = 0$. But $\cos AD = \cos AP \div \cos p$; therefore, &c.

Multiply—(1) by sin BC, and (2) by sin AB, and add the results. Then

$$\sin BC \cos AP + \sin AB \cos CP = \sin AC \cos BP;$$

or writing $\sin AC = -\sin CA$, we have the algebraic sum

$$\sin BC \cos AP + \sin CA \cos BP + \sin AB \cos CP$$

equal to zero.*

41. **Analogous Theorem in Plano.**—By the method similar to that used in Art. 39, we get

$$BC.AP^2 + CA.BP^2 + AB.CP^2 = -BC.CA.AB.$$

(Townsend's *Modern Geometry*, vol. i.)

42. When the point B (fig. 20) bisects the base AC of the triangle ACP, the formula of Art. 40 readily reduces to

$$\cos AP + \cos CP = 2 \cos \tfrac{1}{2} AC \cos BP.$$

Or,

The sum of the cosines of the sides of a triangle is equal to twice the cosine of half the base into cosine of the bisector of the base. [See also Ex. 13, Art. 36.]

Again, when the point B on the base AC is ninety degrees distant from the middle point of AC, we have

$$\cos AP - \cos CP = 2 \sin \frac{AC}{2} \cos BP,$$

an equation which furnishes an obvious solution to the problem—*Given the base and difference of cosines of the sides, find the locus of the vertex.*

---

* Compare this with the more general results of Arts. 132 and 134.

Examples.

1. Having given the base c of a triangle, and $l \cos a \pm m \cos b$, find the locus of the vertex, l and m being constants.

Ans. A circle, the pole of which divides the base internally (or externally) into segments, the sines of which are to one another as l is to m.

NOTE.—Particular cases of this example, viz., when $l = \pm m$, have been considered in Arts. 42 and 36.

2. Having given the base and ratio of cosines of the sides; find the locus of the vertex.

[Let the arc CP (fig. 20) be a quadrant. Then in the triangle ABP, the formula of Art. 40 reduces thus :

$$\frac{\cos AP}{\cos BP} = \frac{\sin AC}{\sin BC};$$

hence, according to the given conditions, C is a fixed point on the base, and the vertex therefore describes a great circle having C for pole. (Cf. Art. 56, Ex. 11.)]

3. Show that the points A, B, C (fig. 20), will lie on the same great circle if

$$\sin BC \sin AL + \sin CA \sin BL + \sin AB \sin CL = 0,$$

where AL, BL, CL, are secondaries to any great circle L.

What is the corresponding theorem *in plano*?

(*Vide* Townsend's *Modern Geometry*, vol. i., chap. 5.)

4. If the base c of a triangle be a quadrant, show that if any point X be taken on it,

$$\cos CX = \cos a \sin AX + \cos b \cos AX.$$

5. From any three points on a great circle, secondaries, x, y, z; x', y', z'; x'', y'', z'', are drawn to the sides of a triangle; prove the determinant relation

$$\begin{vmatrix} \sin x & \sin y & \sin z \\ \sin x' & \sin y' & \sin z' \\ \sin x'' & \sin y'' & \sin z'' \end{vmatrix} = 0.$$

(Apply Ex. 3.)

43. **Theorem III.**—*If from a fixed point P on the surface of a sphere an arc of a great circle be drawn, meeting a small circle at A and B, and if PA and PB be denoted by ρ and ρ', then*

$$\tan \tfrac{1}{2}\rho \tan \tfrac{1}{2}\rho' = \text{constant}.$$

Let C (fig. 21) be the pole of the small circle. Join CA, CB, and CP by arcs of great circles. Let the angle APC be denoted by θ, the arc CP by δ, and the arc CA by a.

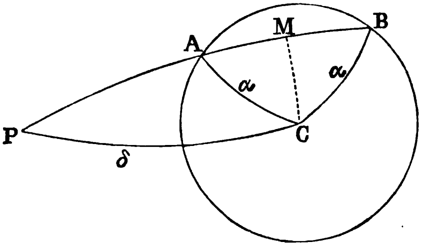

Fig. 21.

From the triangle APC we have

$$\cos a = \cos \rho \cos \delta + \sin \rho \sin \delta \cos \theta.$$

Let $\tan \frac{1}{2}\rho = t$. Then $\cos \rho = \dfrac{1 - t^2}{1 + t^2}$, and $\sin \rho = \dfrac{2t}{1 + t^2}$.

Substituting these values for $\cos \rho$ and $\sin \rho$ in the above equation, and arranging in powers of t, we get

$$t^2(\cos a + \cos \delta) - 2t \sin \delta \cos \theta + \cos a - \cos \delta^* = 0. \quad (1)$$

---

* This may be regarded as the polar equation of the small circle, and its analogue *in plano*, $\rho^2 - 2\rho d \cos \theta + d^2 - a^2 = 0$, can be deduced at once by aid of the formulæ of Art. 18. If for t we substitute its reciprocal $\frac{1}{t}$, we form a new equation, the roots of which are the reciprocals of the roots of equation (1). The product of the roots of the reciprocal equation is clearly constant, and it is therefore the equation of a circle. It therefore follows that if on any radius vector, PA, drawn from any fixed point P to a circle, a point A' be taken, such that $\tan \frac{1}{2}PA \tan \frac{1}{2}PA' =$ constant, then the locus of A' is also a circle. This method of transformation is termed *Spherical Inversion*, and a summary of it will be found in Chapter XII., Section v.

By proceeding in identically the same manner we obtain from the triangle BPC an equation in t' (where $t' = \tan \frac{1}{2}\rho'$) in all respects the same as that in t', and hence it follows that $\tan \frac{1}{2}\rho$ and $\tan \frac{1}{2}\rho'$ are the roots of the above quadratic, and therefore their product,

$$\tan \tfrac{1}{2}\rho \tan \tfrac{1}{2}\rho' = \frac{\cos a - \cos \delta}{\cos a + \cos \delta} = \tan \tfrac{1}{2}(\delta - a) \tan \tfrac{1}{2}(\delta + a), \quad (2)$$

which value is independent of θ, and depends only on the fixed quantities a and δ; therefore, &c.

Otherwise thus:—

Draw CM perpendicular to AB. Then $AM = BM$; and since (Art. 27, Ex. 1) $\cos \delta = \cos PM \cos CM$, and $\cos a = \cos AM \cos CM$, we have, by division,

$$\frac{\cos AM}{\cos PM} = \frac{\cos a}{\cos \delta}.$$

Therefore,

$$\frac{\cos AM - \cos PM}{\cos AM + \cos PM} = \frac{\cos a - \cos \delta}{\cos a + \cos \delta},$$

which reduces at once to

$$\tan \tfrac{1}{2}\rho \tan \tfrac{1}{2}\rho' = \tan \tfrac{1}{2}(\delta - a) \tan \tfrac{1}{2}(\delta + a).$$

Remark.—In fig. 21 it is to be observed that we have *two* triangles, CPA and CPB, having the same *three* parts a, δ, and θ. Whenever more than one triangle can be constructed from three given parts, the solution of the triangle is said to be ambiguous (see Chap. V.) However, it should be noted that the remaining sides ρ and ρ' of the triangles are not independent of each other, but are connected by the relation of this Article.

Examples.

1. Deduce the ***corresponding theorem in plano.***

Ans.—Euclid, III. xxxvi.

2. If the vertical angle C of a triangle be equal to the sum of the base angles, and if the perpendicular p from the vertex on the base divides it into segments c_1 and c_2, show that

$$\tan \tfrac{1}{2} c_1 \tan \tfrac{1}{2} c_2 = \tan^2 \tfrac{1}{2} p.$$

[Let p meet the base in D. Produce CD through D to C', so that $DC' = DC$. Then the triangles ABC and ABC' are equal in all respects, and the vertices of the quadrilateral $ACBC'$ are concyclic, being equidistant from the middle point of AB. Therefore, by Theorem III.,

$$\tan \tfrac{1}{2} c_1 \tan \tfrac{1}{2} c_2 = \tan \tfrac{1}{2} CD \tan \tfrac{1}{2} C'D = \tan^2 \tfrac{1}{2} p.]$$

3. Given the base AB, and that $C = A + B$, find the locus of the vertex C.

Ans.—A circle having its pole at the middle point of the base.

[Note.—In this case the chordal triangle is right-angled at C.]

4. In Theorem III., fig. 21, show that $\tan \frac{1}{2} \rho + \tan \frac{1}{2} \rho'$ is a maximum, when PAB passes through C, and determine the maximum value. (Use the quadratic in t.)

Ans. $\dfrac{2 \sin \delta}{\cos \alpha + \cos \delta}$.

5. Show that $\dfrac{\sin \frac{1}{2} (\rho + \rho')}{\sin \frac{1}{2} \rho \sin \frac{1}{2} \rho'}$, varies as $\cos \theta$, and explain when $\theta = 90°$.

6. If through a point inside a circle two chords be drawn at right angles, and if their segments be ρ_1, ρ_2; ρ_3, ρ_4, show that

$$\tan^2 \tfrac{1}{2} \rho_1 + \tan^2 \tfrac{1}{2} \rho_2 + \tan^2 \tfrac{1}{2} \rho_3 + \tan^2 \tfrac{1}{2} \rho_4 = \text{const.} = \frac{4 \sin^2 \alpha}{(\cos \alpha + \cos \delta)^2}.$$

(*Dub. Univ. Exam. Papers.*)

[Equation (1) gives us

$$\tan \tfrac{1}{2} \rho_1 + \tan \tfrac{1}{2} \rho_2 = \frac{2 \sin \delta \cos \theta}{\cos \alpha + \cos \delta},$$

and for the perpendicular chord

$$\tan \tfrac{1}{2} \rho_3 + \tan \tfrac{1}{2} \rho_4 = \frac{2 \sin \delta \sin \theta}{\cos \alpha + \cos \delta}.$$

Square, add, and reduce by equation (2).]

7. Using the notation of Ex. 6, show that

$$\cot^2 \tfrac{1}{2} \rho_1 + \cot^2 \tfrac{1}{2} \rho_2 + \cot^2 \tfrac{1}{2} \rho_3 + \cot^2 \tfrac{1}{2} \rho_4 = \frac{4 \sin^2 \alpha}{(\cos \alpha - \cos \delta)^2}.$$

8. Describe a circle bisecting the circumferences of three small circles.

TABLE OF FORMULAE.

FORMULAE.	SUPPLEMENTAL FORMULAE.	ANALOGOUS FORMULAE IN PLANO.
$\cos a = \cos b \cos c + \sin b \sin c \cos A$. (Art. 26.) $\cos b = \cos c \cos a + \sin c \sin a \cos B$. ,, $\cos c = \cos a \cos b + \sin a \sin b \cos C$. ,,	$\cos A + \cos B \cos C = \sin B \sin C \cos a$. (Art. 34.) $\cos B + \cos C \cos A = \sin C \sin A \cos b$. ,, $\cos C + \cos A \cos B = \sin A \sin B \cos c$. ,,	$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, (Art. 28.) and $A + B + C = \pi$. (Art. 35.)
$\sin \frac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$. (Art. 33. $\cos \frac{1}{2}A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$. ,, $\tan \frac{1}{2}A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}$. ,, With similar expressions for $\sin \frac{1}{2}B$, etc., and $\sin \frac{1}{2}C$, etc.	$\sin \frac{1}{2}a = \sqrt{-\frac{\cos S \cos(S-A)}{\sin B \sin C}}$. (Art. 36.) $\cos \frac{1}{2}a = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}$. ,, $\tan \frac{1}{2}a = \sqrt{-\frac{\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}}$. ,,	$\sin \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$. $\cos \frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$. $\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$. and $A + B + C = \pi$.

$\sin A = \frac{2n}{\sin b \sin c}$. (Art. 30.) $2n = 2\sqrt{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}$. $= \sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}$.	$\sin a = \frac{2N}{\sin B \sin C}$. (Art. 36.) $2N = 2\sqrt{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)}$ $=\sqrt{1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C}$.	$\sin A = \frac{2\Delta}{bc}$. $2\Delta = 2\sqrt{s(s-a)(s-b)(s-c)}$. $A+B+C=\pi$.
$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2n}{\sin a \sin b \sin c}$. (Art. 31.)	$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \frac{2N}{\sin A \sin B \sin C}$ (Art. 31.)	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\Delta}{abc} = \frac{1}{D}$, Δ being the area of the triangle, and D the diameter of its circumscribed circle.
$\cot a \sin b = \cot A \sin C + \cos b \cos C$. (Art. 37.) $\cot b \sin c = \cot B \sin A + \cos c \cos A$. ,, $\cot c \sin a = \cot C \sin B + \cos a \cos B$. ,,	$\cot a \sin c = \cot A \sin B + \cos c \cos B$. (Art. 38.) $\cot b \sin a = \cot B \sin C + \cos a \cos C$. ,, $\cot c \sin b = \cot C \sin A + \cos b \cos A$. ,,	$\frac{\sin A}{a} = \frac{\sin B}{b} =$ etc. (Art. 39.)

Miscellaneous Examples.

1 If s and s' are the segments of the base made by the perpendicular from the vertex, and σ and σ' those made by the bisector of the vertical angle, show that

$$\tan\frac{s-s'}{2}\tan\frac{\sigma-\sigma'}{2}=\tan^2\frac{a-b}{2}.$$

—(*Dublin Univ. Exam. Papers.*)

[We have

$$\frac{\cos a}{\cos b}=\frac{\cos s}{\cos s'} \text{ and } \frac{\sin a}{\sin b}=\frac{\sin\sigma}{\sin\sigma'}.$$

Taking the difference to the sum in each equation, we get

$$\tan\frac{a-b}{2}\tan\frac{a+b}{2}=\tan\frac{s-s'}{2}\tan\frac{c}{2}, \qquad (1)$$

and

$$\tan\frac{a-b}{2}\cot\frac{a+b}{2}=\tan\frac{\sigma-\sigma'}{2}\cot\frac{c}{2}. \qquad (2)$$

Multiplying (1) and (2) together, we have

$$\tan\frac{s-s'}{2}\tan\frac{\sigma-\sigma'}{2}=\tan^2\frac{a-b}{2}.\ ]$$

2. Prove that

$$\sin b\sin c+\cos b\cos c\cos A=\sin B\sin C-\cos B\cos C\cos a.$$

[If from the extremities of the base BC quadrants CC' and BB' be measured on the sides towards the vertex, the left-hand side of this equation is obviously $\cos B'C'$.

Again, if secondaries be drawn to the sides at C and B, the angle between them is equal to the arc $B'C'$ (Chap. I., Art. 6), and the cosine of this angle is equal to the expression on the right-hand side of the equation by Chap. III., Art. 34; therefore, &c.]

Otherwise thus :—

Multiplying together the equations

$$\cos a=\cos b\cos c+\sin b\sin c\cos A,$$

and

$$\sin B\sin C\cos a-\cos B\cos C=\cos A,$$

and observing that

$$\sin B\sin C\sin^2 a=\sin b\sin c\sin^2 A, \qquad \text{(Art. 31.)}$$

the above result is obtained.

3. Given a, b, and c, the sides of a triangle, find γ, the arc joining the middle points of a and b.

We have

$$\cos\gamma = \cos\frac{a}{2}\cos\frac{b}{2} + \sin\frac{a}{2}\sin\frac{b}{2}\cos C$$

$$= \cos\frac{a}{2}\cos\frac{b}{2} + \frac{\cos c - \cos a\cos b}{4\cos\frac{a}{2}\cos\frac{b}{2}} \qquad \text{(by Art. 26.)}$$

$$= \frac{(1+\cos a)(1+\cos b) + \cos c - \cos a\cos b}{4\cos\frac{a}{2}\cos\frac{b}{2}}$$

$$= \frac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}},$$

with similar expressions for the arcs joining the middle points of b and c, c and a.

Note.—In the colunar triangle

$$\cos\gamma' = \frac{1+\cos c-\cos a-\cos b}{4\sin\frac{a}{2}\sin\frac{b}{2}}.$$

4. If α, β, and γ be the arcs joining the middle points of the sides, prove

$$\frac{\cos\alpha}{\cos\frac{a}{2}} = \frac{\cos\beta}{\cos\frac{b}{2}} = \frac{\cos\gamma}{\cos\frac{c}{2}} = \frac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}} = \sin S;$$

and hence, if one of the arcs be a quadrant, the others are also quadrants. (See Ex. 3, and fig. 38, Art. 104.)

5. If the side c of a spherical triangle be a quadrant, and δ the arc drawn at right angles to it from the opposite vertex, show that

$$\cot^2\delta = \cot^2 A + \cot^2 B.$$

—(*Science and Art Exam. Papers.*)

[Let X be the foot of the perpendicular δ on the base AB; then

$$\tan\delta = \tan A\ .\ \sin AX = \tan B\ .\ \cos AX;$$

therefore, &c.]

6. Given the vertical angle in position and magnitude, and the ratio of the cosines of the base angles, find the locus of the pole of the base.

[Let the perpendicular on the base divide the vertical angle into parts α and β; then by Art. 26, Ex. 6,

$$\frac{\cos A}{\cos B} = \frac{\sin \alpha}{\sin \beta}$$

therefore the perpendicular is fixed in direction, and it obviously contains the pole of the base; therefore, &c.]

7. Prove that in fig. 19,

$$2 \cot XY = \cot AY + \cot BY.^*$$

[For

$$\frac{\sin AY}{\sin BY} = \frac{\sin AX}{\sin XB} = \frac{\sin (AY - XY)}{\sin (XY - BY)};$$

therefore, &c.]

8. If a, b, c, d be the sides of a spherical quadrilateral taken in order, and δ and δ' the diagonals; the arc ϕ joining the middle points of the diagonals is given by the equation

$$\cos \phi = \frac{\cos a + \cos b + \cos c + \cos d}{4 \cos \frac{\delta}{2} \cos \frac{\delta'}{2}},$$

and give the corresponding theorem *in plano*. (See Art. 36, Ex. 14.)

[The sum of the squares of the sides of a quadrilateral = the sum of the squares of its diagonals + four times the square of the line joining the middle points of the diagonals. (See M'Dowell's *Geom. Exercises.*)

9. The arc ϕ' joining the middle points of the opposite sides b and d of a quadrilateral is given by the equation

$$\cos \phi' = \frac{\cos a + \cos c + \cos \delta + \cos \delta'}{4 \cos \frac{b}{2} \cos \frac{d}{2}},$$

---

* An arc AB, divided internally in X and externally in Y, such that $\sin AY : \sin BY :: \sin AX : \sin XB$, is said to be cut harmonically by the segment XY; and the cotangents of either triad of arcs, viz. AY, XY, BY, or AX, AB, AY, measured from either extremity, are therefore in arithmetical progression.

with a similar expression for the arc joining the middle points of a and b. (See Ex. 8.)

10. If the arcs joining the middle points of opposite sides of a quadrilateral be each quadrants, the arc ϕ joining the middle points of the diagonals δ and δ' is given by the equation

$$-\cos\phi = \frac{\cos\delta + \cos\delta'}{2\cos\frac{\delta}{2}\cos\frac{\delta'}{2}}.$$

11. Prove that the angle ϕ between the perpendicular from the vertex on the base and the bisector of the vertical angle is thus given:

$$\tan\phi = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)} \cdot \tan\frac{A-B}{2}.^{*}$$

—(*Dub. Univ. Exam. Papers.*)

[For

$$\frac{\cos A}{\cos B} = \frac{\sin\left(\frac{C}{2}+\phi\right)}{\sin\left(\frac{C}{2}-\phi\right)};$$

therefore, taking the difference to sum,

$$\tan\frac{A+B}{2}\tan\frac{A-B}{2} = \cot\frac{C}{2}\tan\phi.$$

Now substitute the value of $\tan\frac{A+B}{2}$ in Art. 33, Ex. 9; therefore, &c.]

12. Given the base of a triangle and the bisectors of the vertical angle, construct it.

[Using the notation, fig. 19,

$$\cos XY = \cos i \cos\eta;$$

also

$$\cot AX + \cot AY = 2\cot AB, \qquad \text{(Ex. 7.)}$$

two equations which determine the points X and Y.]

---

[*] Tan ϕ is also equal to

$$\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\tan\tfrac{1}{2}(A+B).$$

13. Given the base AB and the sum of the sides of a triangle, find the locus of a point O, where the external bisector of the vertical angle C meets the secondary to BC at the point B. (Cf. Galbraith and Haughton's *Euclid*, I. II. III., Ex. 200.)

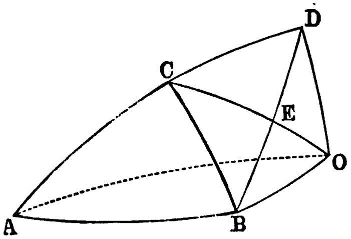

Fig. 22.

[Produce AC (fig. 22) through C to D, such that $CD = BC$. Therefore AD = given sum of the sides.

Since CO bisects the angle BCD, it bisects BD at right angles.

Since BD is bisected at right angles, $BO = OD$, and the triangles CDO and CBO are equal in every respect; hence the angle CDO is a right angle. Therefore

$$\cos AD = \frac{\cos AO}{\cos OD} = \frac{\cos AO}{\cos BO}.$$

In the triangle ABO we have thus given the base AB and ratio of the cosines of the sides, to find the locus of the vertex, for a solution of which see Art. 42, Ex. 2.]

14. Given the base AB and difference of sides, find the locus of the point O, where the secondary to BC at the point B meets the internal bisector of the vertical angle. (See Ex. 13.)

15. Two places on the Earth's surface are distant, one $\theta°$ from the Pole the other $\theta°$ from the Equator, and their difference of longitude is ϕ; find the angular distance between them. —(*Science and Art Exam. Papers.*)

$$\cos \delta = \sin 2\theta \cos^2 \frac{\phi}{2}.$$

16. In an isosceles triangle, if each of the base angles be double the vertical angle, prove that

$$\cos a \cos \frac{a}{2} = \cos \left(c + \frac{a}{2}\right).$$

—(*Lond. Univ. Exam. Papers.*)

[If x denote one of the equal bisectors of the sides, we have

$$\cos x = \cos a \cos \frac{a}{2} + \sin a \sin \frac{a}{2} \cos A,$$

and

$$\cos x = \cos c \cos \frac{a}{2} + \sin c \sin \frac{a}{2} \left(2 \cos^2 \frac{A}{2} - 1\right),$$

where A is the vertical angle.

Again,

$$\frac{\sin a}{\sin c} = 2 \cos A,$$

eliminating $\cos A$; therefore, &c.]

17. If a side c of a spherical triangle be a quadrant, show that

(α). $\cot a \cot b + \cos C = 0.$

(β). $\cos S \cos (S - C) + \cos (S - A) \cos (S - B) = 0.$

—(*Science and Art Exam. Papers.*)

(See Art. 26.) Also

$$\tan \frac{c}{2} = 1;$$

therefore, &c., Art. 36.

18. If λ, μ, and ν be the segments towards the angles of the internal bisectors of the angles of a triangle ABC from the point of concurrence, show that

$$\begin{vmatrix} \cos \lambda & \cos \mu & \cos \nu \\ \sin a & \sin b & \sin c \\ \cos a & \cos b & \cos c \end{vmatrix} = 0.$$

[We have

(1). $\cos \lambda = \cos b \cos \nu + \sin b \sin \nu \cos \frac{C}{2}.$

(2). $\cos \mu = \cos a \cos \nu + \sin a \sin \nu \cos \frac{C}{2}.$

Eliminate $\cos \frac{C}{2}$. Therefore,

$$\cos \lambda \sin a - \cos \mu \sin b = \cos \nu \sin (a - b).$$

Similarly,

$$\cos\mu \sin b - \cos\nu \sin c = \cos\lambda \sin(b - c),$$

and

$$\cos\nu \sin c - \cos\lambda \sin a = \cos\mu \sin(c - a);$$

whence, on adding,

$$\Sigma \cos\lambda \sin(b - c),$$

or its equivalent determinant form, given above, is equal to zero.]

19. If λ, μ, and ν be the segments towards the angles of two external bisectors (of angles B and C), and one internal bisector of the angles of a triangle from the point of concurrence, show that

$$\begin{vmatrix} -\cos\lambda & \cos\mu & \cos\nu \\ \sin a & \sin b & \sin c \\ -\cos a & \cos b & \cos c \end{vmatrix} = 0.$$

20. Using the notation of Example 18, prove that

$$\Sigma^{*} \sin\lambda \sin(b - c) \cos\frac{A}{2} = 0.$$

[Multiplying Ex. 18, (1) by $\cos a$, and (2) by $\cos b$, and subtracting, we get

$$\cos a \cos\lambda - \cos b \cos\mu + \sin(a - b) \sin\nu \cos\frac{C}{2} = 0,$$

which, added to similar equations, involving b and c, and c and a, gives the required result.]

21. Using the notation of Ex. 19, show that

$$\begin{vmatrix} \sin\lambda \cos\frac{A}{2} & \sin\mu \sin\frac{B}{2} & \sin\nu \cos\frac{C}{2} \\ \sin a & \sin b & \sin c \\ -\cos a & \cos b & \cos c \end{vmatrix} = 0.$$

---

[*] The equivalent determinant form is

$$\begin{vmatrix} \sin\lambda \cos\frac{A}{2} & \sin\mu \cos\frac{B}{2} & \sin\nu \cos\frac{C}{2} \\ \sin a & \sin b & \sin c \\ \cos a & \cos b & \cos c \end{vmatrix} = 0.$$

22. If θ, ϕ, and ψ denote the arcs connecting the middle points of pairs of opposite sides and diagonals of a quadrilateral, viz. a and b, c and d, and δ and δ' respectively, prove that

$$\cos\theta\cos\frac{a}{2}\cos\frac{b}{2}+\cos\phi\cos\frac{c}{2}\cos\frac{d}{2}-\cos\psi\cos\frac{\delta}{2}\cos\frac{\delta'}{2}=\tfrac{1}{2}(\cos\delta+\cos\delta').$$

(See Exs. 8 and 9.)

23. A, B, and C are three concyclic points on a sphere; X, Y, and Z are three other points; show that the relation

$$\begin{vmatrix}\cos AX & \cos AY & \cos AZ\\ \cos BX & \cos BY & \cos BZ\\ \cos CX & \cos CY & \cos CZ\end{vmatrix}=0.$$

[By aid of Art. 40 we can eliminate $\sin BC$, $\sin CA$, and $\sin AB$, from three given equations; therefore, &c.]

24. If the sides of a quadrilateral taken in order are a, b, c, d, show that the diagonals δ and δ' intersect at an angle θ, given by the equation

$$\cos\theta=\frac{\cos a\cos c-\cos b\cos d}{\sin\delta\sin\delta'}.$$

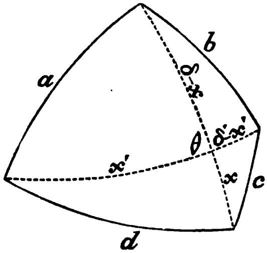

Fig. 23.

[For, representing the segments of the diagonals by x, $\delta-x$, and x', $\delta'-x'$, respectively, we have

$$\cos a=\cos x'\cos(\delta-x)+\sin x'\sin(\delta-x)\cos\theta. \qquad (1)$$
$$\cos c=\cos x\cos(\delta'-x')+\sin x\sin(\delta'-x')\cos\theta. \qquad (2)$$
$$\cos b=\cos(\delta-x)\cos(\delta'-x')-\sin(\delta-x)\sin(\delta'-x')\cos\theta. \qquad (3)$$
$$\cos d=\cos x\cos x'-\sin x\sin x'\cos\theta. \qquad (4)$$

By subtracting the product of equations (3) and (4) from the product of (1) and (2), the above expression for θ is obtained.]

25. By a method similar to that used in Ex. 24, show that if ϕ and ψ are the angles between a and c, b and d, respectively,

$$\cos\psi = \frac{\cos c \cos a - \cos\delta \cos\delta'}{\sin b \sin d}. \qquad (a)$$

$$\cos\phi = \frac{\cos b \cos d - \cos\delta \cos\delta'}{\sin c \sin a}. \qquad (b)$$

26. An arc δ drawn from the vertex of a triangle to meet the base at an angle θ may be expressed by the equation

$$\sin\delta = \frac{\cos a \cos y - \cos b \cos x}{\sin c \cos\theta},$$

where x and y are the segments of the base adjacent to the sides a and b respectively.

[In the figure of Ex. 24 suppose three vertices of the quadrilateral to be concyclic.]

27. In any triangle, prove that

$$\tan\frac{a}{2} : \tan\frac{b}{2} : \tan\frac{c}{2} = \cos(S - A) : \cos(S - B) : \cos(S - C).$$

(See Art. 36 (11.)

28. Prove the following expression* for $\cos S$:—

$$-\cos S = \frac{\sin\frac{1}{2}a \sin\frac{1}{2}b \sin C}{\cos\frac{1}{2}c} = \frac{n}{2\cos\frac{1}{2}a \cos\frac{1}{2}b \cos\frac{1}{2}c}.$$

(Apply Art. 36, (9) and (10).)

Also

$$\sin s = \frac{\cos\frac{1}{2}A \cos\frac{1}{2}B \sin c}{\sin\frac{1}{2}C}.$$

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* The equation for calculating the sum of the angles, having given two sides and the included angle, or, by the aid of Art. 30, the three sides of a spherical triangle, will be afterwards proved as *Cagnoli's Theorem.* (Art. 103.)

29. Prove that

$$\sin(S - A) = \frac{1 + \cos a - \cos b - \cos c}{4 \cos \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}},$$

and similar expressions for $\sin(S - B)$ and $\sin(S - C)$.

[For

$$\sin(S - A) = \sin \frac{B + C}{2} \cos \frac{A}{2} - \cos \frac{B + C}{2} \sin \frac{A}{2};$$

therefore, &c., by the aid of note on Ex. 8, Art. 33.

Or thus:—

Take the colunar triangle, whose angles are $\pi - B$, $\pi - C$, A, and sides $\pi - b$, $\pi - c$, a (Art. 16). Let

$$2S' = \pi - B + \pi - C + A = 2\pi - (B + C - A);$$

hence

$$S' = \pi - (S - A),$$

and the formula adapted to the colunar triangle becomes

$$\sin S' = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}};$$

therefore, &c.] (Cf. Ex. 4.)

30. Prove that

$$\cos(s - a) = \frac{1 + \cos B + \cos C - \cos A}{4 \cos \frac{B}{2} \cos \frac{C}{2} \sin \frac{A}{2}}.$$

31. Prove that

$$\cos(S - A) = \frac{\cos \frac{b}{2} \cos \frac{c}{2}}{\cos \frac{a}{2}} \sin A.$$

32. Prove that

$$\sin(S - A) = \frac{\sin \frac{b}{2} \sin \frac{c}{2} + \cos \frac{b}{2} \cos \frac{c}{2} \cos A}{\cos \frac{a}{2}}.$$

(See Ex. 29.)

[It is evident that the numerator of the fraction is equal to the cosine of the arc joining the middle points of the sides $\pi - b$, $\pi - c$, of the colunar triangle; therefore, &c. (Art. 26, and Ex. 4.)]

33. In any triangle,

$$\frac{\cos a - \cos b}{1 - \cos c} + \frac{\sin (A - B)}{\sin C} = 0.$$

(See Art. 35, Ex. 3.)

34. Given the formula, Art. 36, for $\tan \frac{a}{2}$ prove that

$$-\tan S = \frac{\cot \frac{a}{2} \cot \frac{b}{2} + \cos C.}{\sin C}.$$

35. Show that

$$\cot \frac{a}{2} \cos (S - A) = \cot \frac{b}{2} \cos (S - B) = \cot \frac{c}{2} \cos (S - C)$$

$$= - \cot \frac{a}{2} \cot \frac{b}{2} \cot \frac{c}{2} \cos S = \frac{n}{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c};$$

and hence

$$\Sigma \cos^2 (S - A) = \cos^2 S \Sigma \cot^2 \frac{a}{2} \cot^2 \frac{b}{2}.$$

36. If

$$\tan \frac{a}{2} \tan \frac{b}{2} = \tan^2 \frac{c}{2},$$

show that

$$\cos (S - A) \cos (S - B) = \cos^2 (S - C).$$

37. If a spherical triangle is equal and similar to its polar triangle,

$$\sec a = \sec b \sec c + \tan b \tan c.$$

Give a particular example of such a triangle.

—(*Science and Art Exam. Papers.*)

38. Prove that by a closer approximation, employing the notation used in Legendre's Theorem (Art. 29),

$$\cos A = \cos A' - \frac{\beta\gamma \sin^2 A'}{6r^2} + \frac{\beta\gamma (a^2 - 3\beta^2 - 3\gamma^2) \sin^2 A'}{180r^4}.$$

39. Deduce the following formula for $a+b$ in terms of the opposite angles A and B, and the base c:—

$$\frac{a+b}{2} = \frac{c}{2} + \tan\frac{A}{2}\tan\frac{B}{2}\sin c + \frac{1}{2}\tan^2\frac{A}{2}\tan^2\frac{B}{2}\sin 2c$$
$$+ \frac{1}{3}\tan^3\frac{A}{2}\tan^3\frac{B}{2}\sin 3c + \ldots$$

—(*Bishop Law's Prize Exam., Trin. Coll., Dub.*)

[For
$$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2},$$

(Art. 33, Ex. 9.)

being of the form $\tan\frac{a+b}{2} = n\tan\frac{c}{2}$, can be expanded in series for $\frac{a+b}{2}$.]

—(See Todhunter's *Plane Trig.*, Art. 299.)

40. From any three points on a sphere are drawn to each of three points on a great circle arcs, a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3; making angles with the great circle, respectively, A_1, A_2, A_3, etc. . . . ; prove the determinant relation

$$\begin{vmatrix} \sin a_1\cos A_1 & \sin a_2\cos A_2 & \sin a_3\cos A_3 \\ \sin b_1\cos B_1 & \sin b_2\cos B_2 & \sin b_3\cos B_3 \\ \sin c_1\cos C_1 & \sin c_2\cos C_2 & \sin c_3\cos C_3 \end{vmatrix} = 0.$$

41. If a and b are the sides, and c the base of a triangle, prove that if θ be the angle which β, the bisector of the base, makes with the base,

$$\tan^2\tfrac{1}{2}a\tan^2\tfrac{1}{2}b = \frac{\tan^4\frac{1}{2}c - 2\tan^2\frac{1}{2}c\tan^2\frac{1}{2}\beta\cos 2\theta + \tan^4\frac{1}{2}\beta}{1 - 2\tan^2\frac{1}{2}c\tan^2\frac{1}{2}\beta\cos 2\theta + \tan^4\frac{1}{2}c\tan^4\frac{1}{2}\beta}.$$

—(*Dublin Univ. Exam. Papers.*)

[Find $\tan^2\frac{1}{2}a$, and $\tan^2\frac{1}{2}b$, and multiply the results. See Ex. 3, Paper XII].

42. Find the locus of the vertex, having given the base and—

(1) Ratio of cotangents of the base angles.
(2) Sum or difference of cotangents.
(3) $l\cot A + m\cot B$. (Cf. Art. 39, Ex. 10.)

—(*Educational Times*, April, 1885.)

CHAPTER IV.

RIGHT-ANGLED TRIANGLES.

SECTION I.

44. Geometrical Deduction of Formulæ connecting the Parts of a Right-angled Triangle.—The formulæ necessary for the solution of right-angled triangles have been deduced in Chapter III. as particular cases of the general formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

We, however, add the following geometrical method of obtaining them, as it is both simple and instructive:—

Let *ABC* (fig. 24) be a triangle right-angled at *C*, and let *O* be the centre of the sphere on which it is described. Since *C* is a right angle, the planes *OCA* and *OCB* are perpendicular to each other (Art. 6). Take *R* any point on the radius *OC*, and draw *RP* perpendicular to *OC*, and *RQ* perpendicular to *OB*. Join *PQ*. Since the planes *COA* and *COB* are at right angles, the angle *PRQ* is right, and, as in Art. 32, the angle *OQP* is also right. Hence we have

$$\frac{OQ}{OP} = \frac{OQ}{OR} \cdot \frac{OR}{OP};$$

that is, $\cos c = \cos a \cos b.$ (1)

$$\frac{QR}{QO} = \frac{QR}{QP} \cdot \frac{QP}{QO};$$

that is, $\tan a = \cos B \tan c.$

Similarly, $\tan b = \cos A \tan c.$ (2)

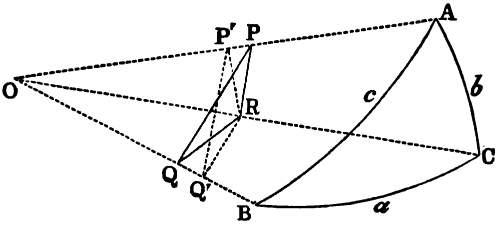

Fig. 24.

$$\frac{PR}{PO} = \frac{PR}{PQ} \cdot \frac{PQ}{PO};$$

that is, $\sin b = \sin B \sin c.$

Similarly, $\sin a = \sin A \sin c.$ (3)

$$\frac{RP}{RO} = \frac{RP}{RQ} \cdot \frac{RQ}{RO};$$

that is, $\tan b = \tan B \sin a.$

Similarly, $\tan a = \tan A \sin b.$ (4)

Again, from R draw RP' perpendicular to OA, and draw RQ' perpendicular to OC, in the plane COB, and meeting OB in Q'. Then it is clear, from the construction, that the plane of the triangle $P'Q'R$ is perpendicular to the line OA, just as the plane of the triangle PQR is perpendicular to OB; and therefore, as in Art. 32, $OP'Q'$ is a right angle, $RP'Q' = A$; and we have

$$\cot A \cot B = \frac{RP'}{RQ'} \cdot \frac{RQ}{RP} = \frac{RP'}{RP} \cdot \frac{RQ}{RQ'}.$$

But $$\frac{RP'}{RP} = \sin RPP' = \cos ROP = \cos b,$$

and $$\frac{RQ}{RQ'} = \sin RQ'Q = \cos ROQ' = \cos a;$$

therefore $$\cot A \cot B = \cos a \cos b = \cos c. \tag{5}$$

$$\frac{\cos B}{\cos b} = \frac{QR}{QP} \div \frac{RP'}{RP} = \frac{QR}{RP'} \cdot \frac{RP}{QP} = \frac{QR}{OR} \cdot \frac{OR}{RP'} \cdot \frac{RP}{OP} \cdot \frac{OP}{QP}$$

$$= \frac{\sin a}{\sin b} \cdot \frac{\sin b}{\sin c} = \frac{\sin a}{\sin c} = \sin A;$$

therefore, Similarly,
$$\left.\begin{aligned} \cos B &= \sin A \cos b. \\ \cos A &= \sin B \cos a. \end{aligned}\right\} \tag{6}$$

45. **Napier's Rules of Circular Parts.**—The formulæ of the foregoing Article enable us, being given any two parts other than the right angle, to determine all the remaining parts of the triangle. They are learned by most students separately and in general terms. Napier, however, has given two rules, called after him, which embrace them all, and which are found by some to be useful aids to the memory; their utility, however, is much questioned by others. We give them here without any investigation, considering it sufficient to merely state them.

The Student may test them by deducing all the preceding formulæ from them, and, if anxious for further information regarding them, Napier's *Mirifici Logarithmorum, Canonis Descriptio*, or Todhunter's *Spherical Trigonometry*, may be consulted.

Omitting the right angle, the two sides which include it and the complements of the remaining parts are termed the *Circular Parts* of the triangle. Thus the five circular parts are a, b, $\frac{1}{2}\pi - A$, $\frac{1}{2}\pi - c$, $\frac{1}{2}\pi - B$. The student may remark that the order in which the circular parts are here written is that in which they naturally occur in the triangle.

Divide a circle (fig. 25) into five sectors, and on each sector write one of the circular parts, taking care to write them round the circle in the order they are written above; that is, in the order in which they occur in the triangle.

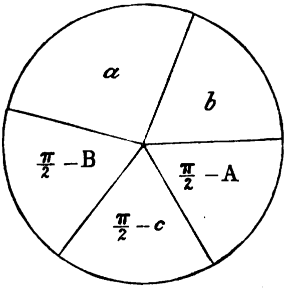

Fig. 25.

Any one of the parts being selected and called the *middle part*, the two parts contiguous to it are called the *adjacent parts*, and the remaining two are called the *opposite parts*. Thus, for example, if $\frac{1}{2}\pi - c$ be selected for middle part, then $\frac{1}{2}\pi - A$ and $\frac{1}{2}\pi - B$ will be the adjacent parts, and a and b will be the opposite parts.

This being premised, Napier's Rules are the following:—

I. Sine (middle part) = product of tangents of adjacent parts.

II. Sine (middle part) = product of cosines of opposite parts.

45 (*a*). **Further Fundamental Relations.**—In addition to the foregoing formulæ the following results, analogous to those of Euc. VI., VIII., are of importance in the theory of right-angled triangles. Let a perpendicular be drawn to the hypotenuse of a right-angled triangle, from the opposite vertex, then the segments of the hypotenuse are connected with the sides, and the perpendicular by the relations.

(1) *The tangent of either side is a geometric mean between the tangent of the adjacent segment and the tangent of the hypotenuse.*

(2) *The sine of the perpendicular is a geometric mean between the tangents of the segments of the hypotenuse.*

For if p be the perpendicular, α and β the segments of the hypotenuse adjacent to the sides a and b, we have, by Art. 44 (2),

$$\cos B = \frac{\tan a}{\tan c} = \frac{\tan \alpha}{\tan a}.$$

Therefore, $\tan^2 a = \tan \alpha \tan c.$

Similarly, $\tan^2 b = \tan \beta \tan c.$

Again, by Art. 44 (4),

$$\tan \alpha = \sin p \tan \widehat{pa},$$

and $$\tan \beta = \sin p \tan \widehat{pb}.$$

But $\widehat{pa}$, the angle between p and a, is the complement of $\widehat{pb}$. Therefore, by multiplication, we obtain

$$\tan \alpha \tan \beta = \sin^2 p.$$

45 (*b*). **Quadrantal Triangles.**—When one side of a triangle is a quadrant it is termed a *Quadrantal Triangle.* The polar reciprocal of such a triangle is obviously right-angled, and the solution of the quadrantal is thus reduced at once to that of the right-angled triangle. Hence, the parts of a quadrantal triangle are connected by the following formulæ, supplemental to those of Art. 44:—

(1) $\cos C = -\cos A \cos B,$

(2) $\tan A = -\cos b \tan C,$

(3) $\sin A = \sin a \sin C,$

(4) $\tan A = \tan a \sin B,$

(5) $\cos C = -\cot a \cot b,$

(6) $\cos a = \cos A \sin b.$

Examples.

1. Show that formula (1), Art. 44, reduces to Euc. I. XLVII. when the radius of the sphere is infinitely great.

[This appears at once on substituting the formulæ of Art. 18, and reducing.]

2. Show that formula (5) reduces to an identity *in plano*. Similarly, also formula (6) reduces to an identity.

3. Formulæ (2), (3), and (4) reduce to the ordinary expressions for the cosine, sine, and tangent of a plane angle respectively.

4. If c be less than 90°, show that a and b are of like affection.*

[In this case $\cos c$ is positive; and hence $\cos a$ and $\cos b$ must have like signs, as appears from formula (1).]

5. If c be greater than 90°, a and b are of opposite affection.

6. The sides of a right-angled triangle must be each less than a quadrant, or two of them must be each greater than a quadrant.

[This follows on combining Examples 4 and 5.]

7. Show that a side and the hypotenuse of a right-angled triangle are of the same or opposite affection, according as the included angle is less than, or greater than, a right angle.

[Apply formula (2), Art. 44.]

8. The sides of a right-angled triangle are of the same affection as the opposite angles.

[By formula (6), $\cos a = \dfrac{\cos A}{\sin B}$; and therefore $\cos a$ is positive or negative, according as $\cos A$ is positive or negative.]

9. If one of the sides of a right-angled triangle be equal to the opposite angle, the remaining parts are each equal to 90°.

[See Art. 36, Ex. 9.]

10. If the angle A of a right-angled triangle be acute, show that the difference of the sides which contain it is less than 90°.

[The opposite side (a) is less than 90°, but it is greater than the difference between the other two; therefore $c - b < 90°$.]

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* Two angles are said to be of the same affection when they are each less than, or each greater than, a right angle.

SECTION II.

Solution of Right-angled Triangles—Numerical Examples.

46. **Solution of Triangles.**—It has been already stated that every spherical triangle has six *parts*—namely, the three angles and the three sides, and that when any three of the parts are given, the remaining three can be determined.

The process by which the remaining three are determined is called the *solution* of the triangle.

In the practical solution of triangles the numerical calculations are performed by aid of a table of logarithms. Hence, if we have at our disposal two or more formulæ, from which any required part can be determined in terms of other known parts, we should use that formula which is most readily adapted to logarithmic computation. If it should happen that a required part can only be obtained from a formula not adapted to logarithmic computation, it can be found by throwing the formula into logarithmic form by the use of a subsidiary angle. Any notice of this process is postponed, however, for the present. (See Art. 76.)

We now proceed to the solution of all the cases which can be presented. They are six in number, and, as before remarked, we require to know only two parts, the right angle forming a constant known part.

Remark.—In determining any of the parts, the use of the parts previously determined should be carefully avoided, lest they should be erroneous. Each part should be determined from the given parts alone.

It is always well to test the accuracy of the results by using the parts obtained to determine one of the given parts.

47. CASE I.—*Having given the two sides a and b,* to determine A, B, and c, we have—

For A, $\cot A = \cot a \sin b.$ [Formula (4), Art. 44.]

For B, $\cot B = \cot b \sin a.$,, ,, ,,

For c, $\cos c = \cos a \cos b.$,, (1), ,,

In this case the solution is unique, as there is manifestly no ambiguity, none of the parts being determined from their sines.

Examples.

1. Given $a = 54^\circ\ 16'$ and $b = 33^\circ\ 12'$.

For A, $10 + L \cot A = L \cot (54^\circ\ 16') + L \sin (33^\circ\ 12')$;

therefore $L \cot A = 9{\cdot}5954382,$

and $A = 68^\circ\ 29'\ 53''.$

For B, $10 + L \cot B = L \cot (33^\circ\ 12') + L \sin (54^\circ\ 16')$;

therefore $L \cot B = 10{\cdot}0935879,$

and $B = 38^\circ\ 52'\ 26''.$

For c, $10 + L \cos c = L \cos a + L \cos b$;

therefore $L \cos c = 9{\cdot}6890261,$

and $c = 60^\circ\ 44'\ 46''.$

2. Given $a = 55^\circ\, 18'$ and $b = 39^\circ\, 27'$.

$$L \cot A = 9{\cdot}6434280\,; \quad \therefore\ A = 66^\circ\, 15'\ \ 6''.$$
$$L \cot B = 9{\cdot}9996157\,; \quad \therefore\ B = 45^\circ\ \ 1'\, 31''.$$
$$L \cos c = 9{\cdot}6430438\,; \quad \therefore\ c = 63^\circ\, 55'\, 21''.$$

3. Given $a = 56^\circ\, 20'$ and $b = 78^\circ\, 40'$.

$$L \cot A = 9{\cdot}8149722\,; \quad \therefore\ A = 56^\circ\, 51'\ \ 7''.$$
$$L \cot B = 9{\cdot}2222192\,; \quad \therefore\ B = 80^\circ\, 31'\, 48''.$$
$$L \cos c = 9{\cdot}0371914\,; \quad \therefore\ c = 83^\circ\, 44'\, 44\tfrac{1}{2}''.$$

48. CASE II.—*Having given the two angles A and B*, to determine a, b, and c, we have—

For a, $\cos A = \cos a \sin B$. [Formula (6), Art. 44.]
For b, $\cos B = \cos b \sin A$. ,, ,, ,,
For c, $\cos c = \cot A \cot B$. ,, (5), ,,

In this case, as in Case I., there is no ambiguity.

Examples.

1. Given $A = 74^\circ\, 15'$ and $B = 32^\circ\, 10'$.

For a,
$$L \cos a = 10 + L \cos (74^\circ\, 15') - L \sin (32^\circ\, 10')$$
$$= 9{\cdot}7074497\,;$$
therefore $a = 59^\circ\, 20'\, 44''$.

For b,
$$L \cos b = 10 + L \cos (32^\circ\, 10') - L \sin (74^\circ\, 15')$$
$$= 9{\cdot}9442480\,;$$
therefore $b = 28^\circ\, 24'\, 54''$.

For c,
$$10 + L \cos c = L \cot A + L \cot B.$$
$$L \cos c = 9{\cdot}6516976\,;$$
therefore $c = 63^\circ\, 21'\, 24\tfrac{1}{2}''$.

2. Given $A = 52° 26'$ and $B = 49° 15'$.

$L \cos a = 9{\cdot}9056850$; $\therefore a = 36° 24' 34{\cdot}5''$.

$L \cos b = 9{\cdot}9156750$; $\therefore b = 34° 33' 40''$.

$L \cos c = 9{\cdot}8213599$; $\therefore c = 48° 29' 20''$.

3. Given $A = 64° 15'$ and $B = 48° 24'$.

$L \cos a = 9{\cdot}7641507$; $\therefore a = 54° 28' 53''$.

$L \cos b = 9{\cdot}8675405$; $\therefore b = 42° 30' 47''$.

$L \cos c = 9{\cdot}6316912$; $\therefore c = 64° 38' 38''$.

49. Case III.—*Having given the hypotenuse c and a side a,* to determine A, B, and b, we have—

For A, $\sin A = \sin a \div \sin c$. [Formula (3), Art. 44.]

For B, $\cos B = \tan a \div \tan c$. ,, (2), ,,

For b, $\cos b = \cos c \div \cos a$. ,, (1), ,,

In this case there is an apparent ambiguity in the value of A, but this is removed by the consideration that A and a are always of the same affection. (See Ex. 8, Art. 45).

Examples.

1. Given $c = 72° 30'$ and $a = 45° 15'$.

For A, $L \sin A = 10 + L \sin (45° 15') - L \sin (72° 30')$

$= 9{\cdot}8719522$;

therefore $A = 48° 7' 44{\cdot}5''$.

For B, $L \cos B = L \tan (45° 15') - L \tan (72° 30') + 10$

$= 9{\cdot}5025123$;

therefore $B = 71° 27' 15''$.

For b, $L \cos b = L \cos (72° 30') - L \cos (45° 15') + 10$

$= 9{\cdot}6305601$;

therefore $b = 64° 42' 52''$.

2. Given $c = 76^\circ\, 42'$ and $a = 47^\circ\, 18'$.

$L \sin A = 9{\cdot}8780442;\ \therefore\ A = 49^\circ\ 2'\, 24{\cdot}5''.$

$L \cos B = 9{\cdot}4085340;\ \therefore\ B = 75^\circ\ 9'\, 24{\cdot}75''.$

$L \cos b = 9{\cdot}5304897;\ \therefore\ b = 70^\circ\, 10'\, 13''.$

3. Given $c = 91^\circ\, 18'$ and $a = 72^\circ\, 27'$.

$L \sin A = 9{\cdot}9794116;\ \therefore\ A = 72^\circ\, 29'\, 48''.$

$L \cos$ sup. of $B = 8{\cdot}8558531;\ \therefore\ B = 94^\circ\ 6'\, 53{\cdot}3''.$

$L \cos$ sup. of $b = 8{\cdot}8764415;\ \therefore\ b = 94^\circ\, 18'\, 53{\cdot}8''.$

50. Case IV.—*Having given the hypotenuse c and an angle A*, to determine a, b, and B, we have—

For a, $\sin a = \sin A \sin c$. [Formula (3), Art. 44.]
For b, $\tan b = \cos A \tan c$. ,, (2), ,,
For B, $\cot B = \tan A \cos c$. ,, (5), ,,

The apparent ambiguity in the value of a is removed, as in Case III., by the consideration that a and A are of the same affection.

Examples.

1. Given $c = 84^\circ\, 20'$ and $A = 35^\circ\, 25'$.

For a, $L \sin a = L \sin (35^\circ\, 25') + L \sin (84^\circ\, 20') - 10$

$= 9{\cdot}7609396;$

therefore $a = 35^\circ\, 13'\, 4''.$

For b, $L \tan b = L \cos (35^\circ\, 25') + L \tan (84^\circ\, 20') - 10$

$= 10{\cdot}9145116;$

therefore $b = 83^\circ\, 3'\, 29''.$

For B, $L \cot B = L \tan (35^\circ\, 25') + L \cos (84^\circ\, 20') - 10$

$= 8{\cdot}8464279;$

therefore $B = 85^\circ\, 59'\, 1''.$

2. Given $c = 67^\circ\,54'$ and $A = 43^\circ\,28'$.

$L \sin a = 9\cdot8044046$; $\therefore\ a = 39^\circ\,35'\,51''$.

$L \tan b = 10\cdot2522138$; $\therefore\ b = 60^\circ\,46'\,25\tfrac{1}{2}''$.

$L \cot B = 9\cdot5521908$; $\therefore\ B = 70^\circ\,22'\,24''$.

3. Given $c = 22^\circ\,18'\,30''$ and $A = 47^\circ\,39'\,36''$.

$L \sin a = 9\cdot4480545$; $\therefore\ a = 16^\circ\,17'\,41''$.

$L \tan b = 9\cdot4414574$; $\therefore\ b = 15^\circ\,26'\,53''$.

$L \cot B = 10\cdot0065973$; $\therefore\ B = 44^\circ\,33'\,53\cdot4''$.

51. CASE V.—*Having given a side a and the adjacent angle B*, to determine A, b, and c, we have—

For A, $\cos A = \cos a \sin B$. [Formula (6), Art. 44.]

For b, $\tan b = \tan B \sin a$. „ (4), „

For c, $\cot c = \cot a \cos B$. „ (2), „

In this case there is evidently no ambiguity.

Examples.

1. Given $a = 31^\circ\,20'\,45''$ and $B = 55^\circ\,30'\,30''$.

For A, $L \cos A = L \cos (31^\circ\,20'\,45'') + L \sin (55^\circ\,30'\,30'') - 10$

$= 9\cdot8475168$;

therefore $A = 45^\circ\,15'\,30\cdot6''$.

For b, $L \tan b = L \tan (55^\circ\,30'\,30'' + L \sin (31^\circ\,20'\,45'') - 10$

$= 9\cdot8791734$;

therefore $b = 37^\circ\,7'\,50''$.

For c, $L \cot c = L \cot (31^\circ\,20'\,45'') + L \cos (55^\circ\,30'\,30'') - 10$

$= 9\cdot9683434$;

therefore $c = 47^\circ\,5'\,11''$.

2. Given $a = 82^\circ\, 6'$ and $B = 43^\circ\, 28'$.

$$L \cos A = 8{\cdot}9756733;\ \therefore A = 84^\circ\, 34'\, 28''.$$
$$L \tan b = 9{\cdot}9726026;\ \therefore b = 43^\circ\, 11'\, 38''.$$
$$L \cot c = 9{\cdot}0030707;\ \therefore c = 84^\circ\, 14'\, 57''.$$

3. Given $a = 42^\circ\, 30'\, 30''$ and $B = 53^\circ\, 10'\, 30''$.

$$L \cos A = 9{\cdot}7709180;\ \therefore A = 53^\circ\, 50'\, 12''.$$
$$L \tan b = 9{\cdot}9554001;\ \therefore b = 42^\circ\, 3'\, 47''.$$
$$L \cot c = 9{\cdot}8155179;\ \therefore c = 56^\circ\, 49'\, 8''.$$

52. Case VI.—*Having given a side a and the opposite angle A*, to determine b, c, and B, we have—

For b, $\sin b = \tan a \cot A$. [Formula (4), Art. 44.]

For c, $\sin c = \sin a \div \sin A$. „ (3), „

For B, $\sin B = \cos A \div \cos a$. „ (6), „

In this case an ambiguity arises as the parts are determined from their sines, and two values for each are in general admissible. However, for each value of b there will, in general, be only one value for c, since c and b are connected by the relation $\cos c = \cos a \cos b$; and at the same time only one admissible value for B, since $\cos c = \cot A \cot B$. Hence there will in general be only *two* triangles having the given parts, except when the side a is a quadrant and the angle A also 90°, in which case the solution becomes *indeterminate.*

When $a = A$, the formulæ, and also the figure, show that b, c, and B are right angles, and the solution is therefore *unique.*

It is easily seen that a is less than A when both are acute (they must be of the same affection), and greater than A when both are obtuse; and when the data do not satisfy these conditions the solution is impossible.

It appears at once from the figure that there are in general two solutions, for the *colunar* triangle $A'BC$ has also the given parts A' and a, and therefore satisfies the given conditions.

Examples.

1. Given $a = 46^\circ\ 45'$ and $A = 59^\circ\ 12'$.

For b,
$$L \sin b = L \tan (46^\circ\ 45') + L \cot (59^\circ\ 12') - 10$$
$$= 9{\cdot}8018795;$$

therefore $b = 39^\circ\ 19'\ 23\tfrac{1}{2}''$, or $140^\circ\ 40'\ 36\tfrac{1}{2}''$.

For c,
$$L \sin c = L \sin a - L \sin A + 10$$
$$= 9{\cdot}9283797;$$

therefore $c = 57^\circ\ 59'\ 29''$, or $122^\circ\ 0'\ 31''$.

For B,
$$L \sin B = L \cos A - L \cos a + 10$$
$$= 9{\cdot}8734997;$$

therefore $B = 48^\circ\ 21'\ 28''$, or $131^\circ\ 38'\ 32''$.

2. Given $a = 21^\circ\ 39'$ and $A = 42^\circ\ 10'\ 10''$.

$L \sin b = 9{\cdot}6417030$; $\therefore b = 25^\circ\ 59'\ 27{\cdot}8''$, or $154^\circ\ 0'\ 32{\cdot}2''$.

$L \sin c = 9{\cdot}7400177$; $\therefore c = 33^\circ\ 20'\ 13{\cdot}4''$, or $146^\circ\ 39'\ 46{\cdot}6''$.

$L \sin B = 9{\cdot}9016853$; $\therefore B = 52^\circ\ 23'\ 2{\cdot}8''$, or $127^\circ\ 36'\ 57{\cdot}2''$.

3. Given $a = 25^\circ\ 18'\ 45''$ and $A = 15^\circ\ 58'\ 15''$.

[*Ans.* Impossible by Art. 52.]

SECTION III.

Theorems.

53. In this section we propose to investigate some theorems which apply to spherical triangles in general, but which are immediately deducible from a knowledge of the properties of right-angled triangles.

54. **Theorem.**—*The arcs drawn from the vertices of a spherical triangle perpendicular to the opposite sides are concurrent.*

Let AP, BQ (fig. 26), be arcs drawn from A, B, perpendicular to BC and CA, respectively, and let O be their point of intersection.

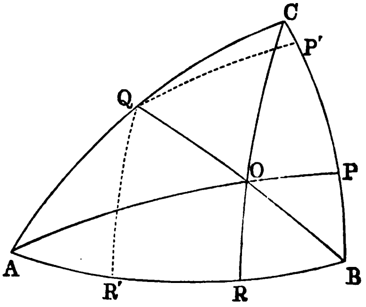

Fig. 26.

Now,
$$\frac{\sin OQ}{\sin OR} = \frac{\sin OAQ}{\sin OAR} = \frac{\sec B}{\sec C};$$

[Formula (6), Art. 44.]

and it is obvious that the sines of the perpendiculars from any other point of AP on the sides b and c bear the same ratio to each other.

Similarly,
$$\frac{\sin OP}{\sin OR} = \frac{\sec A}{\sec C};$$

and hence, on dividing this equation by the former, we get
$$\frac{\sin OP}{\sin OQ} = \frac{\sec A}{\sec B},$$

and therefore the arc joining C to O is perpendicular to the side AB. Thus the perpendiculars meet in a point; and if α, β, γ be the lengths intercepted on them between their point of concurrence and the sides a, b, and c, respectively, we have
$$\sin \alpha : \sin \beta : \sin \gamma = \sec A : \sec B : \sec C.$$

In the same manner, it may be shown that the arcs joining the vertices to the middle points of the opposite sides are concurrent; and if α', β', γ' be the perpendiculars from their point of concurrence on the sides a, b, c, we have
$$\sin \alpha' : \sin \beta' : \sin \gamma' = \operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C.$$

Similarly for the point of intersection of perpendiculars at middle points of sides—
$$\sin \alpha'' : \sin \beta'' : \sin \gamma'' = \sin (S - A) : \sin (S - B) : \sin (S - C.)$$

55. *If from three points A, B, C on a great circle perpendiculars AA', BB', CC' be drawn to another great circle—*

(1) $\sin BC \sin AA' + \sin CA \sin BB' + \sin AB \sin CC' = 0,$
(2) $\sin B'C' \tan AA' + \sin C'A' \tan BB' + \sin A'B' \tan CC' = 0,$
(3) $\sin BC \cot A + \sin CA \cot B + \sin AB \cot C = 0,$
(4) $\sin B'C' \cos A + \sin C'A' \cos B + \sin A'B' \cos C = 0,$

where A, B, C are the angles the perpendiculars AA', BB', CC' make with the great circle ABC.

To prove (1)—

$$\frac{\sin AA'}{\sin AD} = \frac{\sin BB'}{\sin BD} = \frac{\sin CC'}{\sin CD} = \sin D \text{ (fig. 27).} \qquad (\alpha)$$

But since A, B, C are concyclic (Art. 12, Ex. 15),

$$\sin BC \sin AD + \sin CA \sin BD + \sin AB \sin CD = 0$$

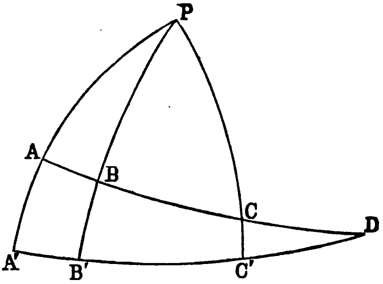

Fig. 27.

Substituting in this expression the values of $\sin AD$, $\sin BD$, and $\sin CD$, obtained from the equations (α), we obtain (1). [Cf. Art. 40.]

To prove (2)—

$$\frac{\tan AA'}{\sin A'D} = \frac{\tan BB'}{\sin B'D} = \frac{\tan CC'}{\sin C'D} = \tan D. \qquad (\beta)$$

But, as before,

$$\sin B'C' \sin A'D + \sin C'A' \sin B'D + \sin A'B' \sin C'D = 0.$$

Substituting in this expression the values of sin $A'D$, sin $B'D$, and sin $C'D$, obtained from the equations (β), we obtain (2).

To prove (3)—

$$\frac{\cos AD}{\cot A} = \frac{\cos BD}{\cot B} = \frac{\cos CD}{\cot C} = \cot D.$$

But

$$\sin BC \cos AD + \sin CA \cos BD + \sin AB \cos CD = 0;$$

therefore, &c. [See Chap. III., Misc. Ex. 42.]

To prove (4)—

$$\frac{\cos A}{\cos A'D} = \frac{\cos B}{\cos B'D} = \frac{\cos C}{\cos C'D} = \sin D.$$

[Formula (6), Art. 44.]

But

$$\sin B'C' \cos A'D + \sin C'A' \cos B'D + \sin A'B' \cos C'D = 0;$$

therefore, &c.

By observing that AA', BB', and CC' meet at P, the pole of $A'B'C'$, we are enabled to write down the following corollaries:—

Cor. 1.—If B be a point on the base AC of a triangle APC, we have

$$\sin BC \cos AP + \sin CA \cos BP + \sin AB \cos CP = 0.$$

[Cf. Art. 40.]

Cor. 2.—

$$\cot AP \sin BPC + \cot BP \sin CPA + \cot CP \sin APB = 0.$$

[Cf. Art. 39, Ex. 10.]

Cor. 3.—

$$\cos A \sin BPC + \cos B \sin CPA + \cos C \sin APB = 0.$$

[Cf. Art. 35, Ex. 5.]

NOTE.—It may be noticed that (1) and (4) are supplemental theorems, as are also (2) and (3). When an arc is drawn from the vertical angle of a triangle to meet the base, (1) connects it with the segments of the base; (2) connects it with the segments of the vertical angle; (3) connects the angle it makes with the base with the segments of the base; and (4) connects the angle it makes with the base with the segments of the vertical angle, the parts of the triangle being supposed known.

56. **Geometrical Deduction of the Analogies of Napier and Delambre.**—*Construction.* On the base AB of the triangle ABC (fig. 28) construct an isosceles triangle AOB, having each base angle $= \frac{1}{2}(A+B)$. From its vertex O draw OX, OY, OZ perpendiculars to the sides a, b, and c of the given triangle. Join OC.

Now the triangles BOX and AOY are equal in all respects; for the side BO and angle X are equal to the side AO and the angle Y, respectively; and the angle $OBX = B - \frac{1}{2}(A+B) = \frac{1}{2}(B-A) =$ angle OAY. Therefore the triangles are equal in all respects, and as a consequence the triangles COX and COY are equal in all respects.

Hence, $$AY = BX = \tfrac{1}{2}(a+b),$$

and $$CX = CY = \tfrac{1}{2}(a-b).$$

But since the angles AOY and BOX are equal, it follows that the angles AOB and XOY are equal, and hence their halves AOZ and COY are equal.

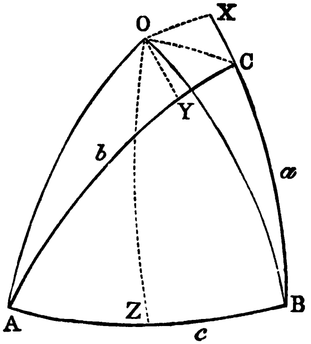

Fig. 28.

Firstly—*Napier's Analogies**:

1. Comparing the triangles AOY and COY, we have

$$\frac{\cos AO}{\cos CO} = \frac{\cos AY}{\cos CY}; \qquad \text{[Art. 44, (1).]}$$

and comparing the triangles, AOZ and COY, we have, also,

$$\frac{\cos AO}{\cos CO} = \frac{\cot OAZ}{\cot OCY}. \qquad \left[\begin{array}{c}\text{Art. 44 (5)}\\ \text{Since } AOZ = COY.\end{array}\right]$$

Therefore,

$$\frac{\tan OAZ}{\tan OCY} = \frac{\cos CY}{\cos AY},$$

---

* This method of obtaining Napier's Analogies is suggested by, and strictly analogous to, the geometrical method of arriving at the formula $\frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{a-b}{a+b}$ in *Plane Trigonometry*. The same principle is used in the *Encyc. Brit.*, Article, "Trigonometry."

or
$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C.$$

2. $\tan OY = \tan OAY \sin AY = \tan OCY \sin CY.$ [Art. 44 (4).]

Hence,
$$\tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \cot \tfrac{1}{2} C.$$

3. $$\tan AO = \frac{\tan AZ}{\cos OAZ} = \frac{\tan AY}{\cos OAY}.$$ [Art. 44 (2).]

Hence
$$\tan \tfrac{1}{2}(a+b) = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c.$$

4. Since $COY = AOZ$, we have [Art. 44 (4).]
$$\frac{\tan CY}{\sin OY} = \frac{\tan AZ}{\sin OZ}.$$

Therefore,
$$\tan CY = \frac{\sin OY}{\sin OZ} \tan AZ = \frac{\sin OAY}{\sin OAZ} \tan AZ,$$
or
$$\tan \tfrac{1}{2}(a-b) = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \tan \tfrac{1}{2} c.$$

Secondly—*Delambre's** *Analogies.*

1. $$\frac{\cos OAZ}{\cos OCY} = \frac{\cos OZ}{\cos OY}$$ [Art. 44 (6).]
$$= \frac{\cos AY}{\cos AZ}.$$ [Art. 44 (1).]

---

* Delambre's Analogies are commonly called Gauss's Formulæ. (See foot note, Art. 33.)

These results may likewise be arrived at by taking O on the other side of the base as the vertex of an isosceles triangle AOB, whose base angles are each $\frac{1}{2}\pi - \frac{1}{2}(A+B)$. Cf. Art. 19 (12).

Therefore

$$\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.$$

.2 $$\frac{\cos OAY}{\cos OCY}=\frac{\sin AOY}{\sin COY} \qquad \text{[Art. 44 (6).]}$$

$$=\frac{\sin AOY}{\sin AOZ}=\frac{\sin AY}{\sin AZ}. \qquad \text{[Art. 44 (3).]}$$

Therefore

$$\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.$$

3. $\cos AOZ=\cos AZ\sin OAZ=\cos CY\sin OCY.$

[Art. 44 (6).]

Hence

$$\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C.$$

4. $$\frac{\sin OAY}{\sin OCY}=\frac{\sin OC}{\sin OA}=\frac{\sin CY}{\sin AZ}.$$

[Art. 44 (3).]

Hence

$$\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C.$$

Examples on Fundamental Formulæ (1) to (6).

1. Prove that, if $C=90°$,

$$\tan\frac{B}{2}=\frac{\sin(s-a)}{\sin s}. \qquad \text{[Cf. Art. 33, Ex. 4.]}$$

2. Prove that—

(*a*) $2n=\sin a\sin b.$ [Art. 30 (3).]

(*b*) $2N=\sin a\sin B.$ [Art. 36 (12).]

(*c*) $\dfrac{n}{N}=\sin c.$ [Art. 36, Ex. 2.]

(*d*) $$\frac{n^2}{N^2}=\frac{\sin^2 a\sin^2 b\sin^2 c}{\sin^2 a+\sin^2 b-\sin^2 c}=\frac{\sin^2 a+\sin^2 b-\sin^2 c}{\sin^2 A-\cos^2 B}.$$

[Cf. Art. 36, Ex. 7.]

3. Prove that

$$\sin^2 a \sin^2 b = \sin^2 a + \sin^2 b - \sin^2 c. \qquad \text{[Art. 44 (1).]}$$

4. Prove that

$$\tan^2 \frac{A}{2} = \frac{\sin (c - b)}{\sin (c + b)}.$$

[This is obviously another form of (2), Art. 44.]

5. What is the Great Circle course and distance between two places in latitude 45° North, with 90° difference of longitude?

Ans. Course makes $\cos^{-1} \frac{1}{\sqrt{3}}$ with meridian. Distance = 60° or $4166\frac{2}{3}$ miles.

6. If p be the perpendicular from the right angle on the hypotenuse, prove that—

(*a*) $\cos^2 p = \cos^2 A + \cos^2 B.$ [Art. 44 (6).]

(*b*) $\cot^2 p = \cot^2 a + \cot^2 b.$ [Art. 44 (4).]

(*c*) $\sin^2 p \sin^2 c = \sin^2 a + \sin^2 b - \sin^2 c.$ [See Ex. 3.]

7. Express the angle A in terms of—(1) b and c; (2) c and a; (3) a and b.

8. Show that

$$2 \sin^2 \tfrac{1}{2} c = \sin^2 \tfrac{1}{2}(a + b) + \sin^2 \tfrac{1}{2}(a - b).$$

[Reduces to (1), Art. 44.]

9. In a spherical triangle, if $c = 90°$, prove that

$$\tan a \tan b + \sec C = 0.$$

[Supplemental to (5), Art. 44.]

10. If the side c of a spherical triangle be a quadrant, show that

$$\sin^2 p = \cot \theta \cot \phi,$$

where p is the perpendicular on c, and θ and ϕ are the segments of the vertical angle.

11. Show that the ratio of the cosines of the segments of the base made by the perpendicular from the vertex is equal to the ratio of the cosines of the sides.

[Apply (1), Art. 44.]

12. Given the base and the ratio of the cosines of the sides of any spherical triangle; find the locus of the vertex, and state the analogue *in plano*.

[See Ex. 11.]

13. A station is in latitude λ (North), and longitude l (East); find the longitudes of the places on the Equator distant δ from the station.

—[*Science and Art Exam. Papers.*]

Ans. Longitudes $= l \pm \cos^{-1}\left(\dfrac{\cos\delta}{\cos\lambda}\right)$.

14. Given the base and base angles of any spherical triangle; calculate the segments x and $c - x$ of the base made by the perpendicular on it from the vertex. *Ans.* $\tan A \sin x = \tan B \sin (c - x)$.

15. What is the shortest distance between two places on the Earth's surface—one in latitude 45° North and longitude 45° West; the other in latitude 45° South and longitude 45° East?

Ans. 120° or 8333·3 miles.

16. Given in a right-angled triangle the vertex fixed, and

$$\cot^2 a + \cot^2 b = \text{const.};$$

find the envelope of the base. [See Ex. 6 (*b*).]

17. Show that the ratio of the cosines of the base angles is equal to the ratio of the sines of the segments of the vertical angle made by the perpendicular drawn from it to the opposite side.

18. Given the vertical angle and ratio of cosines of the base angles; find the envelope of the base.

Ans. A fixed point 90° from the vertex; its position being obtained from Ex. 17.

19. If $A = 36°$ and $B = 60°$, show that—

(1). $a + b + c = 90°$.
(2). The perpendicular on the hypotenuse $= 18°$.
(3). $\cot a - \cot b = 1$.

20. If λ and δ be perpendiculars from a point S on two great circles intersecting at an angle ω, and if the intercepts they make on them from their point of intersection be l and a, show that

$$\cos\omega = \frac{\sin\lambda \tan l + \sin\delta \tan a}{\sin\lambda \tan a + \sin\delta \tan l}.$$

—[Professor Haughton, *Educational Times.*]

[Apply the formulæ of Art. 37 to the triangle SPK, where P and K are the poles of the great circles, and eliminate $\sin\omega$.]

21. Using the notation of Ex. 20, show that—

(1).
$$\frac{\cos \frac{1}{2}(\omega - \omega')}{\cos \frac{1}{2}(\omega + \omega')} = \frac{\tan \frac{1}{2}\lambda \tan \frac{1}{2}l + \tan \frac{1}{2}\delta \tan \frac{1}{2}\alpha}{\tan \frac{1}{2}l \tan \frac{1}{2}\delta + \tan \frac{1}{2}\lambda \tan \frac{1}{2}\alpha}.$$

(2).
$$\frac{\sin \frac{1}{2}(\omega - \omega')}{\sin \frac{1}{2}(\omega + \omega')} = \frac{\tan \frac{1}{2}\delta \tan \frac{1}{2}\lambda - \tan \frac{1}{2}l \tan \frac{1}{2}\alpha}{1 - \tan \frac{1}{2}\alpha \tan \frac{1}{2}l \tan \frac{1}{2}\delta \tan \frac{1}{2}\lambda},$$

ω' being the angle between δ and λ.

Miscellaneous Examples.

In the following Examples the triangle is supposed right-angled (at C), unless otherwise stated:—

1. If β be the bisector of the hypotenuse of a right-angled triangle, show that

$$\sin^2 \beta = \frac{\sin^2 a + \sin^2 b}{4 \cos^2 \frac{c}{2}}.$$

[By Art. 42—

$$\cos a + \cos b = 2 \cos \beta \cos \frac{c}{2}.$$

Squaring, we get—

$$\cos^2 a + \cos^2 b + 2 \cos c = 2(1 - \sin^2 \beta)(1 + \cos c); \; \therefore \; \&c.]$$

2. If ι and η be the internal and external bisectors of C, prove that—

(*a*) $2 \cot^2 \iota = (\cot a + \cot b)^2$,

(*b*) $2 \cot^2 \eta = (\cot a - \cot b)^2$. [See Art. 39, Ex. 1.]

3. Show that—

(*a*) $\cot^2 \iota + \cot^2 \eta = \cot^2 a + \cot^2 b$,

(*b*) $\cot^2 \iota - \cot^2 \eta = 2 \cot a \cot b$.

4. If the bisectors of the vertical angle C meet the base in X and Y, show that

$$2 \cos XY = (\cot^2 a - \cot^2 b) \sin \iota \sin \eta.$$

[Cf. Art. 39, Ex. 6.]

5. If ϕ be the angle between the bisector of the vertical angle and the perpendicular, show that

$$\tan \phi = \frac{\sin (a - b)}{\sin (a + b)}.$$

[See Chap. III., Misc. Ex. 11.]

6. In a right-angled triangle, show that

$$\sin(A+B) = \frac{\cos a + \cos b}{1 + \cos a \cos b}.$$

[Apply Delambre's Formulæ.]

7. Show that

$$\tan S = \cot\frac{a}{2}\cot\frac{b}{2}.$$

Here $$\tan S = \frac{1 + \tan\frac{A+B}{2}}{1 - \tan\frac{A+B}{2}}.$$

[Apply Napier's Analogies.]

8. Construct a right-angled triangle, being given the hypotenuse and—

(1) The sum of the base angles;
(2) The difference of the base angles.

[Apply Gauss's Formulæ.]

9. Given the hypotenuse and the sum or difference of the sides; construct the triangle. (See Ex. 8.)

10. Given the sum of the sides a and b, and the sum of the base angles; solve the triangle.

[Apply Gauss's Formulæ.]

11. Show that

$$\sin\frac{A}{2} = \frac{\sqrt{\sin c + \sin a} + \sqrt{\sin c - \sin a}}{2\sqrt{\sin c}}.$$

[Here, if we divide out by $\sqrt{\sin c}$, and put $\frac{\sin a}{\sin c} = \sin A$, the right-hand side becomes $\frac{1}{2}(\sqrt{1+\sin A} + \sqrt{1-\sin A})$. But $1 \pm \sin A = \left(\sin\frac{A}{2} \pm \cos\frac{A}{2}\right)^2$; therefore, &c.]

12. If α, α', β, β', γ, γ', be the segments of the sides of any spherical triangle made by the perpendiculars from the opposite vertex, show that

$$\cos\alpha\cos\beta\cos\gamma = \cos\alpha'\cos\beta'\cos\gamma'.$$

[For $$\frac{\cos\alpha}{\cos\alpha'} = \frac{\cos a}{\cos b}; \quad \frac{\cos\beta}{\cos\beta'} = \frac{\cos b}{\cos c}; \quad \frac{\cos\gamma}{\cos\gamma'} = \frac{\cos c}{\cos a}.$$

(See Art. 56, Ex. 11). Hence $\frac{\cos\alpha\cos\beta\cos\gamma}{\cos\alpha'\cos\beta'\cos\gamma'} = 1$; therefore, &c.]

13. Using the notation of the preceding Example, show that—

$$(a)\quad \sin\alpha \sin\beta \sin\gamma = \sin\alpha' \sin\beta' \sin\gamma',$$

$$(b)\quad \tan\alpha \tan\beta \tan\gamma = \tan\alpha' \tan\beta' \tan\gamma'.$$

[(*a*) follows from the formula $\tan p = \tan A \sin\alpha = \tan B \sin\alpha'$.]

14. Find the distance between any two points on the Earth's surface, given their latitudes and longitudes. (*London Univ. Exam. Papers.*)

If λ and λ' be their latitudes, and l and l' their longitudes, we have

$$\cos\delta = \sin\lambda \sin\lambda' + \cos\lambda \cos\lambda' \cos(l - l').$$

15. The cosines of the base angles of any triangle are proportional to the cosines of angles which the sides make with the arc joining the vertex to the pole of the perpendicular from the vertex on the base.

[For these latter cosines are equal to the sines of the segments of the vertical angle made by the perpendicular.]

16. If δ be the length of the arc through the vertex of an isosceles triangle, dividing the base into segments α and β, prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \tan\frac{a+\delta}{2}\tan\frac{a-\delta}{2},$$

where a is one of the equal sides of the triangle.

[See Art. 43, or apply Napier's Analogies.]

17. Find a relation connecting the mutual distances of any three places with their latitudes.

[By the aid of Ex. 14. See also Art. 55 (1).]

18. On the Earth considered as a sphere a great circle is drawn, inclined to the Equator at an angle of 30°, and cutting it in points whose longitudes are 0° and 180°. Find the longitudes of the meridians, dividing the great circle into eight equal parts.

19. Two points A and B are taken on two secondaries to a great circle $A'B'$. If θ be the angle between the arcs AB and $A'B'$, its value is found by the equation

$$\sin A'B' = \frac{\sin AB \cos\theta}{\cos AA' \cos BB'}. \quad (\textit{Q. U. I. Exam. Papers.})$$

$$\left[\frac{\cos\theta}{\cos AA' \cos BB'} = \frac{\sin A}{\cos BB'} = \frac{\sin A}{\sin BC} = \frac{\sin A'B'}{\sin AB}; \text{ therefore, \&c.}\right]$$

CHAPTER V.

OBLIQUE-ANGLED TRIANGLES.

57. In the present Chapter we propose to consider the different cases which occur in the solution of oblique-angled triangles. As in right-angled triangles (Chap. IV.) there are six distinct cases, when we are given three of the parts, and are required to determine the remaining three.

Before proceeding to the general solutions of the triangles, we may notice that from particular values of the given parts the values of the other parts may be made to depend on the solutions of right-angled triangles, as for example—

α. Let the side a be a quadrant.

Here the supplemental triangle is right-angled; therefore, &c.

β Let the sides a and b be equal.

The triangle is divided into two equal right-angled triangles by the perpendicular to the base passing through the vertex; therefore, &c.

γ. The general solution of the equilateral triangle may be derived by aid of the *logarithmic* equation

$$\frac{\tan \frac{1}{2} a}{\tan a} = \cos A, \text{ or } 2 \cos A = \sec^2 \tfrac{1}{2} a.$$

58. The six cases (Art. 57) which present themselves are the following :—

I. Having given three sides, a, b, c;
II. Having given three angles, A, B, C;
III. Having given two sides and the included angle, a, C, b;
IV. Having given two angles and the adjacent side, A, c, B;
V. Having given two sides and an angle opposite either, a, b, A;
VI. Having given two angles and a side opposite either, A, B, a;

but these are immediately resolved into *three* distinct problems by the aid of the polar triangle.

For, when three sides are given, and the angles of the triangle are required, the data applied to the polar triangle transforms the problem into a supplemental problem, viz.—Having given three angles of a triangle, find the sides.

Similarly, Cases III. and IV. are supplemental, likewise V. and VI.

59. When numerical values are assigned to the given parts, in order to ascertain the remaining parts in degrees, minutes, and seconds, we employ formulæ adapted to logarithmic computation.

All the formulæ given in Chap. III. are either in logarithmic form or can immediately be thrown into the desired shape by means of a *subsidiary angle.* For the present we select the logarithmic formulæ, and apply them to the following numerical examples, while we reserve the remaining formulæ for future discussion. (See Art. 76.)

60. Case I.—*Having given the three sides a, b, c.* Here we have (Art. 33),

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}$$

$$= \frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

The part under the radical, being a symmetric function of the sides, appears on the right-hand side of the equations for determining A, B, C, and when once calculated is utilized in the calculation of each angle.

The logarithmic form of the above equation is

$$L\tan\frac{A}{2} = 10 - L\sin(s-a) + \tfrac{1}{2}\{L\sin(s-a) + L\sin(s-b)$$
$$+ L\sin(s-c) - L\sin s\}.$$

Examples.

(1). Given $a = 46° 24'$, $b = 67° 14'$, $c = 81° 12'$.

$L\sin s \quad = L\sin 97° 25' = 9{\cdot}9963513.$

$L\sin(s-a) = L\sin 51° 1' = 9{\cdot}8906049.$

$L\sin(s-b) = L\sin 30° 11' = 9{\cdot}7013681.$

$L\sin(s-c) = L\sin 16° 13' = 9{\cdot}4460251.$

$L\tan\frac{1}{2}A = 9{\cdot}6302185;\ \therefore A = 46° 13' 30''.$

$L\tan\frac{1}{2}B = 9{\cdot}8194553;\ \therefore B = 66° 50' 20''.$

$L\tan\frac{1}{2}C = 10{\cdot}0747984;\ \therefore C = 99° 49' 10''.$

(2). Given $a = 108° 14'$, $b = 75° 29'$, $c = 56° 37'$.

$$
\begin{aligned}
L \sin s &= L \sin 120° \ 10' = 9{\cdot}9367988.\\
L \sin (s - a) &= L \sin 11° \ 56' = 9{\cdot}3154947.\\
L \sin (s - b) &= L \sin 44° \ 41' = 9{\cdot}8470714.\\
L \sin (s - c) &= L \sin 63° \ 33' = 9{\cdot}9519799.
\end{aligned}
$$

$$
\begin{aligned}
L \tan \tfrac{1}{2} A &= 10{\cdot}2733789; \ \therefore A = 123° \ 53' \ 47''.\\
L \tan \tfrac{1}{2} B &= 9{\cdot}7418022; \ \therefore B = 57° \ 46' \ 56''.\\
L \tan \tfrac{1}{2} C &= 9{\cdot}6368937; \ \therefore C = 46° \ 51' \ 51{\cdot}5''.
\end{aligned}
$$

(3). Given $a = 57° 17'$, $b = 20° 39'$, $c = 76° 22'$.

$$
\begin{aligned}
L \sin s &= L \sin 77° \ 9' = 9{\cdot}9889849.\\
L \sin (s - a) &= L \sin 19° \ 52' = 9{\cdot}5312649.\\
L \sin (s - b) &= L \sin 56° \ 30' = 9{\cdot}9211066.\\
L \sin (s - c) &= L \sin 0° \ 47' = 8{\cdot}1358104.
\end{aligned}
$$

$$
\begin{aligned}
L \tan \tfrac{1}{2} A &= 9{\cdot}2683336; \ \therefore A = 21° \ 1' \ 2''.\\
L \tan \tfrac{1}{2} B &= 8{\cdot}8784919; \ \therefore B = 8° \ 38' \ 46''.\\
L \tan \tfrac{1}{2} C &= 10{\cdot}6637881; \ \therefore C = 155° \ 31' \ 36{\cdot}5''.
\end{aligned}
$$

(4). Given $a = 68° 45'$, $b = 53° 15'$, $c = 46° 30'$.

$$
\begin{aligned}
L \sin s &= L \sin 84° \ 15' = 9{\cdot}9978093.\\
L \sin (s - a) &= L \sin 15° \ 30' = 9{\cdot}4268988.\\
L \sin (s - b) &= L \sin 31° \ 0' = 9{\cdot}7118393.\\
L \sin (s - c) &= L \sin 37° \ 45' = 9{\cdot}7869056.
\end{aligned}
$$

$$
\begin{aligned}
L \tan \tfrac{1}{2} A &= 10{\cdot}0370184; \ \therefore A = 94° \ 12' \ 40''.\\
L \tan \tfrac{1}{2} B &= 9{\cdot}7520779; \ \therefore B = 58° \ 5' \ 10''.\\
L \tan \tfrac{1}{2} C &= 9{\cdot}6770116; \ \therefore C = 50° \ 50' \ 52\tfrac{1}{2}''.
\end{aligned}
$$

(5). Given $a = 63° 54'$, $b = 47° 18'$, $c = 53° 26'$.

$$
\begin{aligned}
L \sin s &= L \sin 82° \ 19' = 9{\cdot}9960834.\\
L \sin (s - a) &= L \sin 18° \ 25' = 9{\cdot}4995840.\\
L \sin (s - b) &= L \sin 35° \ 1' = 9{\cdot}7587717.\\
L \sin (s - c) &= L \sin 28° \ 53' = 9{\cdot}6839720.
\end{aligned}
$$

$L \tan \frac{1}{2} A = 9 \cdot 9735382$; $\therefore A = 86° 30' 40''$.

$L \tan \frac{1}{2} B = 9 \cdot 7143505$; $\therefore B = 54° 46' 14''$.

$L \tan \frac{1}{2} C = 9 \cdot 7891502$; $\therefore C = 63° 12' 55\frac{1}{2}''$.

61. CASE II.—*Having given the three angles A, B, C.* Here we have (Art. 36),

$$\tan \frac{a}{2} = \sqrt{\frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}$$

$$= \cos (S - A) \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}},$$

with similar values for $\tan \frac{b}{2}$ and $\tan \frac{c}{2}$.

Examples.

(1). Given $A = 68° 30'$, $B = 74° 20'$, $C = 83° 10'$.

$L (-\cos S) = L \cos 67° \ 0' = 9 \cdot 5918780$.

$L \cos (S - A) = L \cos 44° 30' = 9 \cdot 8532421$.

$L \cos (S - B) = L \cos 38° 40' = 9 \cdot 8925365$.

$L \cos (S - C) = L \cos 29° 50' = 9 \cdot 9382576$.

$L \tan \frac{a}{2} = 9 \cdot 8071630$; $\therefore a = 65° 21' 22\frac{1}{2}''$.

$L \tan \frac{b}{2} = 9 \cdot 8464574$; $\therefore b = 70° \ 9' \ 9\frac{1}{2}''$.

$L \tan \frac{c}{2} = 9 \cdot 8921785$; $\therefore c = 75° 55' \ 9''$.

(2). Given $A = 86° 20'$, $B = 76° 30'$, $C = 94° 40'$.

$L (-\cos S) = L \cos 51° 15' = 9 \cdot 7965212$.

$L \cos (S - A) = L \cos 42° 25' = 9 \cdot 8682088$.

$L \cos (S - B) = L \cos 52° 15' = 9 \cdot 7869056$.

$L \cos (S - C) = L \cos 34° \ 5' = 9 \cdot 9181475$.

$\tan \frac{1}{2} a = 9{\cdot}9798385$; $\therefore a = 87^\circ\ 20'\ 28''$.
$\tan \frac{1}{2} b = 9{\cdot}8985353$; $\therefore b = 76^\circ\ 44'\ 2\frac{1}{2}''$.
$\tan \frac{1}{2} c = 10{\cdot}0297772$; $\therefore c = 93^\circ\ 55'\ 31''$.

(3). Given $A = 96^\circ\ 45'$, $B = 108^\circ\ 30'$, $C = 116^\circ\ 15'$.

$L\,(-\cos S) = L \cos 19^\circ\ 45' = 9{\cdot}9750129$.
$L \cos (S - A) = L \cos 64^\circ\ 0' = 9{\cdot}6418420$.
$L \cos (S - B) = L \cos 52^\circ\ 15' = 9{\cdot}7869056$.
$L \cos (S - C) = L \cos 44^\circ\ 30' = 9{\cdot}9532421$.

$\tan \frac{1}{2} a = 9{\cdot}9883536$; $\therefore a = 88^\circ\ 27'\ 49''$.
$\tan \frac{1}{2} b = 10{\cdot}1334172$; $\therefore b = 107^\circ\ 19'\ 52''$.
$\tan \frac{1}{2} c = 10{\cdot}1997537$; $\therefore c = 115^\circ\ 28'\ 13\frac{1}{4}''$.

(4). Given $A = 78^\circ\ 30'$, $B = 118^\circ\ 40'$, $C = 93^\circ\ 20'$.

$L\,(-\cos S) = L \cos 34^\circ\ 45' = 9{\cdot}9146852$.
$L \cos (S - A) = L \cos 66^\circ\ 45' = 9{\cdot}5963154$.
$L \cos (S - B) = L \cos 26^\circ\ 35' = 9{\cdot}9514757$.
$L \cos (S = C) = L \cos 51^\circ\ 55' = 9{\cdot}7901493$.

$\tan \frac{1}{2} a = 9{\cdot}8846878$; $\therefore a = 74^\circ\ 57'\ 46''$.
$\tan \frac{1}{2} b = 10{\cdot}2398481$; $\therefore b = 120^\circ\ 8'\ 49''$.
$\tan \frac{1}{2} c = 10{\cdot}0785217$; $\therefore c = 100^\circ\ 18'\ 11\frac{3}{4}''$.

(5). Given $A = 57^\circ\ 50'$, $B = 98^\circ\ 20'$, $C = 63^\circ\ 40'$.

$L\,(-\cos S) = L \cos 70^\circ\ 5' = 9{\cdot}5323123$.
$L \cos (S - A) = L \cos 52^\circ\ 5' = 9{\cdot}7885323$.
$L \cos (S - B) = L \cos 11^\circ\ 35' = 9{\cdot}9910637$.
$L \cos (S - C) = L \cos 46^\circ\ 15' = 9{\cdot}8398004$.

$\tan \frac{1}{2} a = 9{\cdot}7449903$; $\therefore a = 58^\circ\ 8'\ 19''$.
$\tan \frac{1}{2} b = 9{\cdot}9475217$; $\therefore b = 83^\circ\ 5'\ 36''$.
$\tan \frac{1}{2} c = 9{\cdot}7962584$; $\therefore c = 64^\circ\ 3'\ 20''$.

$L \tan \frac{1}{2} A = 9{\cdot}9735382$; $\therefore A = 86^\circ\ 30'\ 40''$.

$L \tan \frac{1}{2} B = 9{\cdot}7143505$; $\therefore B = 54^\circ\ 46'\ 14''$.

$L \tan \frac{1}{2} C = 9{\cdot}7891502$; $\therefore C = 63^\circ\ 12'\ 55\frac{1}{2}''$.

61. Case II.—*Having given the three angles A, B, C.* Here we have (Art. 36),

$$\tan \frac{a}{2} = \sqrt{\frac{-\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}}$$

$$= \cos (S-A) \sqrt{\frac{-\cos S}{\cos (S-A) \cos (S-B) \cos (S-C)}},$$

with similar values for $\tan \frac{b}{2}$ and $\tan \frac{c}{2}$.

Examples.

(1). Given $A = 68^\circ\ 30'$, $B = 74^\circ\ 20'$, $C = 83^\circ\ 10'$.

$L(-\cos S) = L \cos 67^\circ\ 0' = 9{\cdot}5918780$.

$L \cos (S-A) = L \cos 44^\circ\ 30' = 9{\cdot}8532421$.

$L \cos (S-B) = L \cos 38^\circ\ 40' = 9{\cdot}8925365$.

$L \cos (S-C) = L \cos 29^\circ\ 50' = 9{\cdot}9382576$.

$L \tan \frac{a}{2} = 9{\cdot}8071630$; $\therefore a = 65^\circ\ 21'\ 22\frac{1}{2}''$.

$L \tan \frac{b}{2} = 9{\cdot}8464574$; $\therefore b = 70^\circ\ 9'\ 9\frac{1}{2}''$.

$L \tan \frac{c}{2} = 9{\cdot}8921785$; $\therefore c = 75^\circ\ 55'\ 9''$.

(2). Given $A = 86^\circ\ 20'$, $B = 76^\circ\ 30'$, $C = 94^\circ\ 40'$.

$L(-\cos S) = L \cos 51^\circ\ 15' = 9{\cdot}7965212$.

$L \cos (S-A) = L \cos 42^\circ\ 25' = 9{\cdot}8682088$.

$L \cos (S-B) = L \cos 52^\circ\ 15' = 9{\cdot}7869056$.

$L \cos (S-C) = L \cos 34^\circ\ 5' = 9{\cdot}9181475$.

$\tan \frac{1}{2} a = 9{\cdot}9798385$; $\therefore a = 87^\circ\ 20'\ 28''$.

$\tan \frac{1}{2} b = 9{\cdot}8985353$; $\therefore b = 76^\circ\ 44'\ 2\frac{1}{2}''$.

$\tan \frac{1}{2} c = 10{\cdot}0297772$; $\therefore c = 93^\circ\ 55'\ 31''$.

(3). Given $A = 96^\circ\ 45'$, $B = 108^\circ\ 30'$, $C = 116^\circ\ 15'$.

$L(-\cos S) = L \cos 19^\circ\ 45' = 9{\cdot}9750129$.

$L \cos (S - A) = L \cos 64^\circ\ 0' = 9{\cdot}6418420$.

$L \cos (S - B) = L \cos 52^\circ\ 15' = 9{\cdot}7869056$.

$L \cos (S - C) = L \cos 44^\circ\ 30' = 9{\cdot}9532421$.

$\tan \frac{1}{2} a = 9{\cdot}9883536$; $\therefore a = 88^\circ\ 27'\ 49''$.

$\tan \frac{1}{2} b = 10{\cdot}1334172$; $\therefore b = 107^\circ\ 19'\ 52''$.

$\tan \frac{1}{2} c = 10{\cdot}1997537$; $\therefore c = 115^\circ\ 28'\ 13\frac{1}{2}''$.

(4). Given $A = 78^\circ\ 30'$, $B = 118^\circ\ 40'$, $C = 93^\circ\ 20'$.

$L(-\cos S) = L \cos 34^\circ\ 45' = 9{\cdot}9146852$.

$L \cos (S - A) = L \cos 66^\circ\ 45' = 9{\cdot}5963154$.

$L \cos (S - B) = L \cos 26^\circ\ 35' = 9{\cdot}9514757$.

$L \cos (S = C) = L \cos 51^\circ\ 55' = 9{\cdot}7901493$.

$\tan \frac{1}{2} a = 9{\cdot}8846878$; $\therefore a = 74^\circ\ 57'\ 46''$.

$\tan \frac{1}{2} b = 10{\cdot}2398481$; $\therefore b = 120^\circ\ 8'\ 49''$.

$\tan \frac{1}{2} c = 10{\cdot}0785217$; $\therefore c = 100^\circ\ 18'\ 11\frac{3}{4}''$.

(5). Given $A = 57^\circ\ 50'$, $B = 98^\circ\ 20'$, $C = 63^\circ\ 40'$.

$L(-\cos S) = L \cos 70^\circ\ 5' = 9{\cdot}5323123$.

$L \cos (S - A) = L \cos 52^\circ\ 5' = 9{\cdot}7885323$.

$L \cos (S - B) = L \cos 11^\circ\ 35' = 9{\cdot}9910637$.

$L \cos (S - C) = L \cos 46^\circ\ 15' = 9{\cdot}8398004$.

$\tan \frac{1}{2} a = 9{\cdot}7449903$; $\therefore a = 58^\circ\ 8'\ 19''$.

$\tan \frac{1}{2} b = 9{\cdot}9475217$; $\therefore b = 83^\circ\ 5'\ 36''$.

$\tan \frac{1}{2} c = 9{\cdot}7962584$; $\therefore c = 64^\circ\ 3'\ 20''$.

62. CASE III.—*Having given two sides and the included angle, a, C, b.*

This case is somewhat similar to the corresponding one in *Plane Trig.*, the base angles being calculated from two separate equations, one giving half the sum and the other half the difference of those angles. The formulæ to be employed have been arrived at on Art. 32 (Note to Ex. 8). They have, moreover, been demonstrated geometrically from the relations between the sides and angles of certain right-angled triangles. As they are of fundamental importance in the solution of triangles, we shall now proceed by a direct method to obtain the same results.

63. Napier's Analogies:

Let
$$x = \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b};$$
hence,
$$x = \frac{\sin A \pm \sin B}{\sin a \pm \sin b}. \qquad (1)$$
Now,
$$\cos A + \cos B \cos C = \sin B \sin C \cos a = x \sin C \sin b \cos a,$$
and
$$\cos B + \cos C \cos A = \sin C \sin A \cos b = x \sin C \sin a \cos b.$$

By addition,
$$(\cos A + \cos B)(1 + \cos C) = x \sin C \sin (a + b); \qquad (2)$$
by subtraction,
$$(\cos B - \cos A)(1 - \cos C) = x \sin C\ (a - b). \qquad (3)$$

On taking the positive sign in (1), and dividing by (2), we obtain

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin a + \sin b}{\sin (a + b)} \cdot \frac{1 + \cos C}{\sin C};$$

or

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2}. \qquad (\alpha')$$

On taking the negative sign in (1), we obtain, by a similar method,

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{C}{2}. \qquad (\beta')$$

Employing the supplemental formulæ, viz.,

$$\cos a - \cos b \cos c = \sin b \sin c \cos A = \frac{1}{x} \sin B \cos A \sin c,$$

and

$$\cos b - \cos c \cos a = \sin c \sin a \cos B = \frac{1}{x} \sin A \cos B \sin c,$$

we have, as before,

$$x \,.\, (\cos a \pm \cos b)(1 \mp \cos c) = \sin c \sin (B \pm A).$$

Dividing each of these equations by results in equation (1) to eliminate x, we have formulæ supplemental to (α') and (β'), viz.,

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \frac{c}{2}, \qquad (\gamma')$$

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \frac{c}{2}. \qquad (\delta')$$

64. Delambre's (or Gauss's) Analogies:

By the preceding Article, we have

$$\tan \tfrac{1}{2}(A+B) \tan \frac{C}{2} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}.$$

Hence,

$$1 - \tan \tfrac{1}{2}(A+B) \tan \tfrac{1}{2} C = 1 - \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)};$$

or

$$\frac{-\cos S}{\cos \frac{1}{2}(A+B) \cos \frac{1}{2} C} = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2}(a+b)}; \qquad (1)$$

but

$$-\cos S = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b \sin C}{\cos \frac{1}{2} c}.$$

[Art. 43, Ex. 28.]

Substituting this value in (1), we have at once

$$\cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a+b) \sin \tfrac{1}{2} C. \qquad (a)$$

Again,

$$1 + \tan \tfrac{1}{2}(A-B) \tan \tfrac{1}{2} C = 1 + \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}.$$

Hence,

$$\frac{\cos (S-A)}{\cos \frac{1}{2}(A-B) \cos \frac{1}{2} C} = \frac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b}{\sin \frac{1}{2}(a+b)}; \qquad (2)$$

but (Art. 43, Ex. 31)

$$\cos (S-A) = \frac{\cos \frac{1}{2} b \cos \frac{1}{2} c}{\cos \frac{1}{2} a} \sin A = \sin \tfrac{1}{2} a \cos \tfrac{1}{2} c \frac{\sin B}{\sin \frac{1}{2} b}.$$

Substituting this value in (2), we have

$$\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C. \qquad (\beta)$$

Similarly,

$$\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C, \qquad (\gamma)$$

and

$$\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C. \qquad (\delta)$$

65. It will be thus seen that when each of Napier's Analogies is added to or subtracted from unity, the results are easily transformed into Delambre's formulæ.

It is also to be noticed that if the equation

$$\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C$$

be applied to the colunar triangle of parts a, $\pi-b$, $\pi-c$, A, $\pi-B$, $\pi-C$, we get

$$\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C.$$

And by a similar method, the equation

$$\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C$$

gives

$$\cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.$$

Remark.—From the three preceding Articles it has been shown that Napier's four Analogies are reducible, by aid of the polar triangle, to *two independent theorems*, involving five parts of a spherical triangle; and that Gauss's four formulæ are, by aid of the colunar triangle, likewise reducible to two independent theorems.

66. In the solution of triangles, Case III., the formulæ

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C,$$

and

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C,$$

enable us to calculate $\frac{1}{2}(A+B)$, and $\frac{1}{2}(A-B)$, from which we obtain A and B.

The remaining element c is determined from the equation

$$\cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c = \cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C;$$

or, as will be hereafter shown, it may be determined from the formula

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

without having previously determined A or B.

Examples.

(1). Given $a = 43° 18'$, $b = 19° 24'$, $C = 74° 22'$.
Firstly, determine A and B—

$$L \cos \tfrac{1}{2}(a - b) = L \cos 11° 57' = 9{\cdot}9904848.$$
$$L \cos \tfrac{1}{2}(a + b) = L \cos 31° 21' = 9{\cdot}9314605.$$
$$L \cot \tfrac{1}{2} C \quad = L \cot 37° 11' = 10{\cdot}1199969.$$
$$L \tan \tfrac{1}{2}(A + B) = 10{\cdot}1790212.$$

$$L \sin \tfrac{1}{2}(a - b) = 9{\cdot}3160921.$$
$$L \sin \tfrac{1}{2}(a + b) = 9{\cdot}7162243.$$
$$L \tan \tfrac{1}{2}(A - B) = 9{\cdot}7198647.$$

Therefore,

$$\tfrac{1}{2}(A + B) = 56° 29' 17'', \quad \text{and} \quad \tfrac{1}{2}(A - B) = 27° 41' 0\tfrac{1}{2}''.$$

To determine c—

$$L \cos \tfrac{1}{2}(a+b) \quad = L \cos 31^\circ\ 21'\ \ 0'' = 9{\cdot}9314605.$$

$$L \cos \tfrac{1}{2}(A+B) = L \cos 56^\circ\ 29'\ 17'' = 9{\cdot}7420263.$$

$$L \sin \tfrac{1}{2} C \qquad = L \sin 37^\circ\ 11'\ \ 0'' = 9{\cdot}7813010.$$

Hence

$$L \cos \tfrac{1}{2} c = 9{\cdot}9707352, \quad \text{or} \quad \tfrac{1}{2} c = 20^\circ\ 48'\ 54\tfrac{1}{4}''.$$

$$\textit{Ans.}\ A = 84^\circ\ 10'\ 17\tfrac{1}{2}'', \quad B = 28^\circ\ 48'\ 17\tfrac{1}{2}'', \quad c = 41^\circ\ 35'\ 48\tfrac{1}{2}''.$$

(2). $a = 96^\circ\ 24'\ 30''$, $b = 68^\circ\ 27'\ 26''$, $C = 84^\circ\ 46'\ 40''$.

$$L \cos \tfrac{1}{2}(a-b) = L \cos 13^\circ\ 58'\ 32'' = 9{\cdot}9869503.$$

$$L \cos \tfrac{1}{2}(a+b) = L \cos 82^\circ\ 25'\ 58'' = 9{\cdot}1195505.$$

$$L \cot \tfrac{1}{2} C \qquad = L \cot 42^\circ\ 23'\ 20'' = 10{\cdot}0396387.$$

$$L \tan \tfrac{1}{2}(A+B) = 10{\cdot}9070385.$$

$$L \sin \tfrac{1}{2}(a-b) = 9{\cdot}3829313, \quad L \sin \tfrac{1}{2}(a+b) = 9.9962011.$$

$$L \tan \tfrac{1}{2}(A-B) = 9{\cdot}4263689.$$

$$\tfrac{1}{2}(A+B) = 82^\circ\ 56'\ 19\tfrac{3}{4}'', \quad \tfrac{1}{2}(A-B) = 14^\circ\ 56'\ 40\tfrac{1}{2}''.$$

$$L \cos \tfrac{1}{2}(A+B) = 9{\cdot}0896545.$$

$$L \sin \tfrac{1}{2} C \qquad = 9{\cdot}8287624.$$

$$L \cos \tfrac{1}{2} c = 9{\cdot}8586584, \quad \tfrac{1}{2} c = 43^\circ\ 45'\ 48\tfrac{1}{2}''.$$

$$\textit{Ans.}\ A = 97^\circ\ 53'\ 0\tfrac{1}{4}'', \quad B = 67^\circ\ 59'\ 39\tfrac{1}{4}'', \quad c = 87^\circ\ 31'\ 37''.$$

(3). $a = 76^\circ\ 24'\ 40''$, $b = 58^\circ\ 18'\ 36''$, $C = 116^\circ\ 30'\ 28''$.

$$L \cos \tfrac{1}{2}(a-b) = L \cos 9^\circ\ \ 3'\ \ 2'' = 9{\cdot}9945590.$$

$$L \cos \tfrac{1}{2}(a+b) = L \cos 67^\circ\ 21'\ 38'' = 9{\cdot}5853827.$$

$$L \cot \tfrac{1}{2} C \qquad = L \cot 58^\circ\ 15'\ 14'' = 9{\cdot}7914976.$$

$$L \tan \tfrac{1}{2}(A+B) = 10{\cdot}2006739.$$

$$L \sin \tfrac{1}{2}(a-b) = 9{\cdot}1967450, \quad L \sin \tfrac{1}{2}(a+b) = 9{\cdot}9651760.$$

$$L \tan \tfrac{1}{2}(A-B)\ 9{\cdot}0230666.$$

$$\tfrac{1}{2}(A+B) = 57^\circ\ 47'\ 23\tfrac{3}{4}'', \quad \tfrac{1}{2}(A-B) = 6^\circ\ 1'\ 11\tfrac{1}{2}''.$$

$$L \cos \tfrac{1}{2}(A+B) = 9{\cdot}7267476.$$

$$L \sin \tfrac{1}{2} C \qquad = 9{\cdot}9296171.$$

$$\tfrac{1}{2} c = 52^\circ\ 6'\ 43\tfrac{1}{2}''.$$

Ans. $A = 63^\circ\ 48'\ 35\tfrac{1}{4}''$, $B = 51^\circ\ 46'\ 12\tfrac{1}{4}''$, $c = 104^\circ\ 13'\ 27''$.

(4). $a = 86^\circ\ 18'\ 40''$, $b = 45^\circ\ 36'\ 20''$, $C = 120^\circ\ 46'\ 30''$.

$$L \cos \tfrac{1}{2}(a-b) = L \cos 20^\circ\ 21'\ 10'' = 9{\cdot}9720032.$$

$$L \cos \tfrac{1}{2}(a+b) = L \cos 65^\circ\ 57'\ 30'' = 9{\cdot}6100219.$$

$$L \cot \tfrac{1}{2} C \qquad = L \cot 60^\circ\ 23'\ 15'' = 9{\cdot}7546296.$$

$$\tfrac{1}{2}(A+B) = 52^\circ\ 36'\ 4\tfrac{3}{4}''.$$

$$L \sin \tfrac{1}{2}(a-b) = 9{\cdot}5413288, \quad L \sin \tfrac{1}{2}(a+b) = 9{\cdot}9605894.$$

$$\tfrac{1}{2}(A-B) = 12^\circ\ 12'\ 49''.$$

$$L \sin \tfrac{1}{2} C \qquad = 9{\cdot}9392132.$$

$$L \cos \tfrac{1}{2}(A+B) = 9{\cdot}7834444.$$

$$\tfrac{1}{2} c = 54^\circ\ 19'\ 35\tfrac{3}{4}''.$$

Ans. $A = 64^\circ\ 48'\ 53\tfrac{3}{4}''$, $B = 40^\circ\ 23'\ 15\tfrac{3}{4}''$, $c = 108^\circ\ 39\ \ 11\tfrac{1}{3}''$.

(5). $a = 88^\circ\ 24'$, $b = 56^\circ\ 48'$, $C = 128^\circ\ 16'$.

$$L \cos \tfrac{1}{2}(a-b) = L \cos 15^\circ\ 48' = 9{\cdot}9832735.$$

$$L \cos \tfrac{1}{2}(a+b) = L \cos 72^\circ\ 36' = 9{\cdot}4757304.$$

$$L \cot \tfrac{1}{2} C \qquad = L \cot 64^\circ\ \ 8' = 9{\cdot}6856120.$$

$$\tfrac{1}{2}(A+B) = 57^\circ\ 20'\ 27\tfrac{1}{4}''.$$

$$L \sin \tfrac{1}{2}(a-b) = 9{\cdot}4350161, \quad L \sin \tfrac{1}{2}(a+b) = 9{\cdot}9796578.$$

$$\tfrac{1}{2}(A-B) = 7^\circ\ 52'\ 36\tfrac{1}{4}''.$$

$$L \cos \tfrac{1}{2}(A+B) = 9{\cdot}7321029.$$

$$L \sin \tfrac{1}{2} C \qquad = 9{\cdot}9541517.$$

$$\tfrac{1}{2} c = 60^\circ\ 5'\ 26''.$$

Ans. $A = 65^\circ\ 13'\ 3\tfrac{1}{2}''$, $B = 49^\circ\ 27'\ 51''$, $c = 120^\circ\ 10'\ 52''$.

(6). $a = b = 32° 23' 57''$, $C = 66° 49' 17''$.

—(*Science and Art Exam. Papers.*)

$L \cos \frac{1}{2}(a - b) = 10.$

$L \cos \frac{1}{2}(a + b) = L \cos 32° 23' 57'' = 9{\cdot}9265152.$

$L \cot \frac{1}{2} C = L \cot 33° 24' 38\frac{1}{2}'' = 10{\cdot}1806890.$

$\frac{1}{2}(A + B) = 60° 53' 2''.$

$L \sin a = 9{\cdot}7290144.$

$L \sin \frac{1}{2} C = 9{\cdot}7408650.$

$\frac{1}{2} c = 17° 9' 35\frac{1}{2}''.$

Ans. $A = B = 60° 53' 2''$, $c = 34° 19' 11''$.

67. CASE IV.—*Having given two angles and the adjacent side* (A, c, B).

By *Napier's Analogies :*

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \tfrac{1}{2} c,$$

and

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \tfrac{1}{2} c;$$

from which we may determine $\frac{1}{2}(a + b)$ and $\frac{1}{2}(a - b)$, and thence a and b.

The remaining part C is determined by the equation

$$\cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2} C = \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} c.$$

It is evident that the value of C thus found is ambiguous,* inasmuch as the angle is obtained from the *sine;* and the values are, therefore, C and $180 - C$. The equation has, however, been selected as an example, to show that an ambiguity may sometimes be got rid of by attending to the relations which the parts of a triangle bear to each other; and in the following numerical examples the ambiguity is removed by remembering that the greater side is opposite to the greater angle.

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* By using formula γ (or δ), Art. 64, ambiguity is avoided.

Examples.

(1). $A = 68^\circ\ 40'$, $B = 56^\circ\ 20'$, $c = 84^\circ\ 30'$.

$$L \cos \tfrac{1}{2}(A - B) = L \cos\ 6^\circ\ 10' = 9{\cdot}9974797.$$

$$L \cos \tfrac{1}{2}(A + B) = L \cos 62^\circ\ 30' = 9{\cdot}6644056.$$

$$L \tan \tfrac{1}{2} c \qquad = L \tan 42^\circ\ 15' = 9{\cdot}9582465.$$

$$\tfrac{1}{2}(a + b) = 62^\circ\ 55'\ 9''.$$

$$L \sin \tfrac{1}{2}(A - B) = 9{\cdot}0310890, \quad L \sin \tfrac{1}{2}(A + B) = 9{\cdot}9479289.$$

$$\tfrac{1}{2}(a - b) = 6^\circ\ 16'\ 39''.$$

$$L \cos \tfrac{1}{2}(a + b) = 9{\cdot}6582472.$$

$$L \cos \tfrac{1}{2} c \qquad = 9{\cdot}8693597.$$

$$\tfrac{1}{2} C = 48^\circ\ 39'\ 31\tfrac{3}{4}''.$$

Ans. $a = 60^\circ\ 11'\ 48''$, $b = 56^\circ\ 38'\ 30''$, $C = 97^\circ\ 19'\ 3\tfrac{1}{2}''$.

(2). $A = 31^\circ\ 34'\ 26''$, $B = 30^\circ\ 28'\ 12''$, $c = 70^\circ\ 2'\ 3''$.

$$L \cos \tfrac{1}{2}(A - B) = L \cos\ 0^\circ\ 33'\ \ 7'' = 9{\cdot}9999799.$$

$$L \cos \tfrac{1}{2}(A + B) = L \cos 31^\circ\ \ 1'\ 19'' = 9{\cdot}9329656.$$

$$L \tan \tfrac{1}{2} c \qquad = L \tan 35^\circ\ \ 1'\ 1\tfrac{1}{2}'' = 9{\cdot}8455023.$$

$$\tfrac{1}{2}(a + b) = 39^\circ\ 16'\ 4\tfrac{1}{4}''.$$

$$L \sin \tfrac{1}{2}(A - B) = 7{\cdot}9837459, \quad L \sin \tfrac{1}{2}(A + B) = 9{\cdot}7121160.$$

$$\tfrac{1}{2}(a - b) = 0^\circ\ 45'\ 1''.$$

$$L \cos \tfrac{1}{2}(a + b) = 9{\cdot}8888507.$$

$$L \cos \tfrac{1}{2} c \qquad = 9{\cdot}9132738.$$

$$\tfrac{1}{2} C = 65^\circ\ 1'\ 55''.$$

Ans. $a = 40^\circ\ 1'\ 5\tfrac{1}{4}''$, $b = 38^\circ\ 31'\ 3\tfrac{1}{4}''$, $C = 130^\circ\ 3'\ 50''$.

(3). $A = 130° \, 5' \, 22{\cdot}41''$, $B = 32° \, 26' \, 6{\cdot}41''$, $c = 51° \, 6' \, 11{\cdot}6''$.

—(*Science Siz. Exam. Papers.*)

$$L \cos \tfrac{1}{2}(A - B) = L \cos 48° \, 49' \, 38'' = 9{\cdot}8184449.$$

$$L \cos \tfrac{1}{2}(A + B) = L \cos 81° \, 15' \, 44{\cdot}41'' = 9{\cdot}1815879.$$

$$L \tan \tfrac{1}{2} c = L \tan 25° \, 33' \, 5{\cdot}8'' = 9{\cdot}6795022.$$

$$\tfrac{1}{2}(a + b) = 64° \, 14' \, 7''.$$

$$L \sin \tfrac{1}{2}(A - B) = 9{\cdot}8766380, \quad L \sin \tfrac{1}{2}(A + B) = 9{\cdot}9949301.$$

$$\tfrac{1}{2}(a - b) = 20° \, 0' \, 22''.$$

$$L \cos \tfrac{1}{2}(a + b) = 9{\cdot}6381663.$$

$$L \cos \tfrac{1}{2} c = 9{\cdot}9552944.$$

$$\tfrac{1}{2} C = 18° \, 22' \, 43''.$$

Ans. $a = 84° \, 14' \, 29''$, $b = 44° \, 13' \, 45''$, $C = 36° \, 45' \, 26''$.

(4). $A = 96° \, 46' \, 30''$, $B = 84° \, 30' \, 20''$, $c = 126° \, 46'$.

$$L \cos \tfrac{1}{2}(A - B) = L \cos 6° \, 8' \, 5'' = 9{\cdot}9975058.$$

*L cos supplement of $\tfrac{1}{2}(A + B) = L \cos 89° \, 21' \, 35'' = 8{\cdot}0482011$.

$$L \tan \tfrac{1}{2} c = L \tan 63° \, 23' \, 0'' = 10{\cdot}3000526.$$

Supplement of $\tfrac{1}{2}(a + b) = 89° \, 40' \, 38''$.

$$L \sin \tfrac{1}{2}(A - B) = 9{\cdot}0288420, \quad L \sin \tfrac{1}{2}(A + B) = 9{\cdot}9999729.$$

$$\tfrac{1}{2}(a - b) = 12° \, 2' \, 20''.$$

L cos supplement of $\tfrac{1}{2}(a + b) = 7{\cdot}7506455$, $L \cos \tfrac{1}{2} c = 9{\cdot}6512966$.

$$\tfrac{1}{2} C = 62° \, 44' \, 6\tfrac{5}{8}''.$$

Ans. $a = 102° \, 21' \, 42''$, $b = 78° \, 17' \, 2''$, $C = 125° \, 28' \, 13\tfrac{1}{4}''$.

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* When $\tfrac{1}{2}(A + B)$ exceeds a quadrant, by substituting its supplement in the equation, a value is obtained for the supplement of $\tfrac{1}{2}(a + b)$. See *Napier's Analogies*, and *Gauss's Formulæ*, from which it also appears that $\tfrac{1}{2}(A + B)$ and $\tfrac{1}{2}(a + b)$ are of the same affection.

(5). $A = 84^\circ\ 30'\ 20''$, $B = 76^\circ\ 20'\ 40''$, $c = 130^\circ\ 46'$.

$$L \cos \tfrac{1}{2}(A - B) = L \cos 4^\circ\ 4'\ 50'' = 9{\cdot}9988977.$$

$$L \cos \tfrac{1}{2}(A + B) = L \cos 80^\circ\ 25'\ 30'' = 9{\cdot}2209927.$$

$$L \tan \tfrac{1}{2} c = L \tan 65^\circ\ 23'\ 0'' = 10{\cdot}3389566.$$

$$\tfrac{1}{2}(a + b) = 85^\circ\ 37'\ 50\tfrac{1}{4}''.$$

$$L \sin \tfrac{1}{2}(A - B) = 8{\cdot}8522289, \quad L \sin \tfrac{1}{2}(A + B) = 9{\cdot}9939071.$$

$$\tfrac{1}{2}(a - b) = 8^\circ\ 57'\ 2''.$$

$$L \cos \tfrac{1}{2}(a + b) = 8{\cdot}8818753.$$

$$L \cos \tfrac{1}{2} c = 9{\cdot}6196622.$$

$$\tfrac{1}{2} C = 65^\circ\ 25'\ 46\tfrac{3}{4}''.$$

Ans. $a = 94^\circ\ 34'\ 52\tfrac{1}{4}''$, $b = 76^\circ\ 40'\ 48\tfrac{1}{4}''$, $C = 130^\circ\ 51'\ 33\tfrac{1}{2}''$.

68. CASE V.—*Having given two sides and the angle opposite either* (a, b, A).

The angle B is found from the formula

$$\sin B = \frac{\sin b}{\sin a} \sin A.$$

For the angle C,

$$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}(A + B).$$

For the side c,

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a + b).$$

Since B has been found from its sine, it may have two values, viz. B and $180 - B$, and the triangle in the general case will admit of two solutions.

This is known as the *ambiguous case*, a full discussion of which is given in Arts. 70–75.

The following numerical examples will serve to elucidate how two different triangles may be constructed when three parts are given:—

Examples.

(1). $a = 42^\circ\, 45'$, $b = 47^\circ\, 15'$, $A = 56^\circ\, 30'$.

$$L \sin a = L \sin 42^\circ\, 45' = 9{\cdot}8317423.$$

$$L \sin b = L \sin 47^\circ\, 15' = 9{\cdot}8658868.$$

$$L \sin A = L \sin 56^\circ\, 30' = 9{\cdot}9211066.$$

Hence $L \sin B = 9{\cdot}9552511$;

or

$$B = 64^\circ\, 26'\, 4'', \quad \text{or} \quad 115^\circ\, 33'\, 56''.$$

Firstly—take the value of B less than a right angle.

$$L \cos \tfrac{1}{2}(b + a) = L \cos 45^\circ\, 0'\, 0'' = 9{\cdot}8494850.$$

$$L \cos \tfrac{1}{2}(b - a) = L \cos 2^\circ\, 15'\, 0'' = 9{\cdot}9996650;$$

and

$$L \cot \tfrac{1}{2}(B + A) = L \cot 60^\circ\, 28'\, 2'' = 9{\cdot}7532216.$$

Hence $L \tan \tfrac{1}{2} C = 9{\cdot}9034016$;

or

$$\tfrac{1}{2} C = 38^\circ\, 40'\, 47\tfrac{1}{2}''.$$

$$L \cos \tfrac{1}{2}(B + A) = L \cos 60^\circ\, 28'\, 2'' = 9{\cdot}6927772.$$

$$L \cos \tfrac{1}{2}(B - A) = L \cos 3^\circ\, 58'\, 2'' = 9{\cdot}9989581.$$

$$L \tan \tfrac{1}{2}(b + a) = L \tan 45^\circ\, 0'\, 0'' = 10{\cdot}0000000.$$

$$\tfrac{1}{2} c = 26^\circ\, 17'\, 39''.$$

Secondly—take the value of B greater than a right angle. Then

$$L \cot \tfrac{1}{2}(B + A) = L \cot 86^\circ\, 1'\, 58'' = 8{\cdot}8410586;$$

and by aid of the logarithms in the first case we have

$$\tfrac{1}{2} C = 5^\circ\, 35'\, 50\tfrac{1}{4}''.$$

Again,

$$L\cos\tfrac{1}{2}(B+A) = L\cos 86^\circ\ \ 1'\ 58'' = 8{\cdot}8400167.$$

$$L\cos\tfrac{1}{2}(B-A) = L\cos 29^\circ\ 31'\ 58'' = 9{\cdot}9395561.$$

Hence

$$\tfrac{1}{2}c = 4^\circ\ 32'\ 47\tfrac{1}{8}''.$$

Ans.—

$$B = \ 64^\circ\ 25'\ \ 4'',\quad C = 77^\circ\ 21'\ 35'',\quad c = 52^\circ\ 35'\ 18'';$$

or

$$B = 115^\circ\ 33'\ 56'',\quad C = 11^\circ\ 11'\ 40\tfrac{1}{2}''\quad c = \ 9^\circ\ \ 5'\ 34\tfrac{1}{2}''.$$

(2). $a = 62^\circ\ 15'\ 24''$, $b = 103^\circ\ 18'\ 47''$, $A = 53^\circ\ 42'\ 38''$.

$$L\sin a = L\sin 62^\circ\ 15'\ \ 24'' = 9{\cdot}9469638.$$

$$L\sin b = L\sin 76^\circ\ 41'\ \ 13'' = 9{\cdot}9881693.$$

$$L\sin A = L\sin 53^\circ\ 42'\ \ 38'' = 9{\cdot}9063552.$$

$$B = 62^\circ\ 24'\ 24\tfrac{4}{5}'',\quad \text{or}\quad 117^\circ\ 35'\ 35\tfrac{1}{5}''.$$

Firstly—

$$L\cos\tfrac{1}{2}(b+a) \ = L\cos 82^\circ\ 47'\ \ 5\tfrac{1}{2}'' = 9{\cdot}0989736.$$

$$L\cos\tfrac{1}{2}(b-a) \ = L\cos 20^\circ\ 31'\ 41\tfrac{1}{2}'' = 9{\cdot}9715077.$$

$$L\cot\tfrac{1}{2}(B+A) = L\cot 58^\circ\ \ 3'\ 31\tfrac{2}{5}'' = 9{\cdot}7947983.$$

$$\tfrac{1}{2}C = 77^\circ\ 51'\ 35\tfrac{2}{3}''.$$

$$L\cos\tfrac{1}{2}(B+A) = L\cos 58^\circ\ \ 3'\ 31\tfrac{2}{5}'' = \ 9{\cdot}7234966.$$

$$L\cos\tfrac{1}{2}(B-A) = L\cos\ \ 4^\circ\ 20'\ 53\tfrac{2}{5}'' = \ 9{\cdot}9987482.$$

$$L\tan\tfrac{1}{2}(b+a) \ = L\tan 82^\circ\ 47'\ \ 5\tfrac{1}{2}'' = 10{\cdot}8975738.$$

$$\tfrac{1}{2}c = 76^\circ\ 34'\ 47\tfrac{3}{4}''.$$

Secondly—

$$L\cot\tfrac{1}{2}(B+A) = L\cot 85^\circ\ 39'\ 6\tfrac{3}{5}'' = 8{\cdot}8810181.$$

$$\tfrac{1}{2}C = 29^\circ\ 33'\ 5\tfrac{1}{3}''.$$

$$L\cos\tfrac{1}{2}(B+A) = L\cos 85^\circ\ 39'\ \ 6\tfrac{3}{5}'' = 8{\cdot}8797663.$$

$$L\cos\tfrac{1}{2}(B-A) = L\cos 31^\circ\ 56'\ 28\tfrac{3}{5}'' = 9{\cdot}9286983.$$

$$\tfrac{1}{2}c = 35^\circ\ 12'\ 43''.$$

Ans.—

$$B = \ 62^\circ\ 24'\ 24\tfrac{4}{5}'',\quad C = 155^\circ\ 43'\ 11\tfrac{1}{3}'',\quad c = 153^\circ\ \ 9'\ 35\tfrac{1}{2}'';$$

or

$$B = 117^\circ\ 35'\ 35\tfrac{1}{5}'',\quad C = \ 59^\circ\ \ 6'\ 10\tfrac{2}{3}'',\quad c = \ 70^\circ\ 25'\ 26''.$$

(3). $a = 52^\circ\ 45'\ 20''$, $b = 71^\circ\ 12'\ 40''$, $A = 46^\circ\ 22\ \ 10''$.

$B = 59^\circ\ 24'\ 15\tfrac{3}{4}''$, or $120^\circ\ 35'\ 44\tfrac{1}{4}''$.

Firstly—

$$L\cos\tfrac{1}{2}(b+a) = L\cos 61^\circ\ 59'\ \ 0'' = 9{\cdot}6718468.$$
$$L\cos\tfrac{1}{2}(b-a) = L\cos\ 9^\circ\ 13'\ 40'' = 9{\cdot}9943430.$$
$$L\cot\tfrac{1}{2}(B+A) = L\cot 52^\circ\ 53'\ 12\tfrac{7}{8}'' = 9{\cdot}8788970.$$
$$\tfrac{1}{2}C = 57^\circ\ 49'\ 57\tfrac{3}{4}''.$$

$$L\cos\tfrac{1}{2}(B+A) = L\cos 52^\circ\ 53'\ 12\tfrac{7}{8}'' = 9{\cdot}7805983.$$
$$L\cos\tfrac{1}{2}(B-A) = L\cos\ 6^\circ\ 31'\ \ 2\tfrac{7}{8}'' = 9{\cdot}9971842.$$
$$L\tan\tfrac{1}{2}(b+a) = L\tan 61^\circ\ 59'\ \ 0'' = 10{\cdot}2740209.$$
$$\tfrac{1}{2}c = 48^\circ\ 46'\ 39\tfrac{2}{5}''.$$

Secondly—

$$L\cot\tfrac{1}{2}(B+A) = L\cot 83^\circ\ 28'\ 57\tfrac{1}{8}'' = 9{\cdot}0578349.$$
$$\tfrac{1}{2}C = 13^\circ\ 29'\ 57\tfrac{3}{5}''.$$

$$L\cos\tfrac{1}{2}(B+A) = L\cos 83^\circ\ 28'\ 57\tfrac{1}{8}'' = 9{\cdot}0550190.$$
$$L\cos\tfrac{1}{2}(B-A) = L\cos 37^\circ\ \ 6'\ 47\tfrac{1}{8}'' = 9{\cdot}9017013.$$
$$\tfrac{1}{2}c = 14^\circ\ 58'\ 35\tfrac{1}{4}''.$$

Ans.—

$B = 59^\circ\ 24'\ 15\tfrac{3}{4}''$, $C = 115^\circ\ 39'\ 55\tfrac{1}{2}''$, $c = 97^\circ\ 33'\ 18\tfrac{4}{5}''$,

or

$B = 120^\circ\ 35'\ 44\tfrac{1}{4}''$, $C = 26^\circ\ 59'\ 55\tfrac{1}{5}''$, $c = 29^\circ\ 57'\ 10\tfrac{1}{2}''$.

(4) $a = 48^\circ\ 45'\ 40''$, $b = 67^\circ\ 12'\ 20''$, $A = 42^\circ\ 20'\ 30''$.

$B = 55^\circ\ 39'\ 57''$, or $124^\circ\ 20'\ 3''$.

Firstly—

$$L\cos\tfrac{1}{2}(b+a) = L\cos 57^\circ\ 59'\ \ 0'' = 9{\cdot}7244118.$$
$$L\cos\tfrac{1}{2}(b-a) = L\cos\ 9^\circ\ 13'\ 20'' = 9{\cdot}9943498.$$
$$L\cot\tfrac{1}{2}(B+A) = L\cot 49^\circ\ \ 0'\ 13\tfrac{1}{2}'' = 9{\cdot}9391057.$$
$$\tfrac{1}{2}C = 58^\circ\ 17'\ 9''.$$

$$L\cos\tfrac{1}{2}(B+A) = L\cos 49^\circ\ \ 0'\ 13\tfrac{1}{2}'' = 9{\cdot}8169102.$$
$$L\cos\tfrac{1}{2}(B-A) = L\cos\ 6^\circ\ 39'\ 43\tfrac{1}{2}'' = 9{\cdot}9970576.$$
$$L\tan\tfrac{1}{2}(b+a) = L\tan 57^\circ\ 59'\ \ 0'' = 10{\cdot}2029297.$$

$$\tfrac{1}{2}c = 46^\circ\ 34'\ 4\tfrac{4}{5}''.$$

Secondly—

$$L \cot \tfrac{1}{2}(B+A) = L \cot 83^\circ\ 20'\ 16\tfrac{1}{2}'' = 9{\cdot}0674506.$$

$$\tfrac{1}{2}C = 12^\circ\ 16'\ 7\tfrac{1}{2}''.$$

$$L \cos \tfrac{1}{2}(B+A) = L \cos 83^\circ\ 20'\ 16\tfrac{1}{2}'' = 9{\cdot}0645081.$$

$$L \cos \tfrac{1}{2}(B-A) = L \cos 40^\circ\ 59'\ 46\tfrac{1}{2}'' = 9{\cdot}8778046.$$

$$\tfrac{1}{2}c = 13^\circ\ 48'\ 40''.$$

***Ans.*—**

$B = 55^\circ\ 39'\ 57''$, $C = 116^\circ\ 34'\ 18''$, $c = 93^\circ\ 8'\ 9\tfrac{3}{8}''$;

or

$B = 124^\circ\ 20'\ 3''$, $C = 24^\circ\ 32'\ 15''$, $c = 27^\circ\ 37'\ 20''$.

(5). $a = 46^\circ\ 20'\ 45''$, $b = 65^\circ\ 18'\ 15''$, $A = 40^\circ\ 10'\ 30''$.

$$B = 54^\circ\ 6'\ 19'', \text{ or } 125^\circ\ 53'\ 41''.$$

Firstly—

$$L \cos \tfrac{1}{2}(b+a) = L \cos 55^\circ\ 49'\ 30'' = 9{\cdot}7495218.$$

$$L \cos \tfrac{1}{2}(b-a) = L \cos 9^\circ\ 28'\ 45'' = 9{\cdot}9940291.$$

$$L \cot \tfrac{1}{2}(B+A) = L \cot 47^\circ\ 8'\ 24\tfrac{1}{2}'' = 9{\cdot}9675258.$$

$$\tfrac{1}{2}C = 58^\circ\ 27'\ 43''.$$

$$L \cos \tfrac{1}{2}(B+A) = L \cos 47^\circ\ 8'\ 24\tfrac{1}{2}'' = 9{\cdot}8326414.$$

$$L \cos \tfrac{1}{2}(B-A) = L \cos 6^\circ\ 57'\ 54\tfrac{1}{2}'' = 9{\cdot}9967831.$$

$$L \tan \tfrac{1}{2}(b+a) = L \tan 55^\circ\ 49'\ 30'' = 10{\cdot}1681548.$$

$$\tfrac{1}{2}c = 45^\circ\ 15'\ 53''.$$

Secondly—

$$L \cot \tfrac{1}{2}(B+A) = L \cot 83^\circ\ 2'\ 5\tfrac{1}{2}'' = 9{\cdot}0869538.$$

$$\tfrac{1}{2}C = 12^\circ\ 6'\ 26\tfrac{2}{3}''.$$

$$L \cos \tfrac{1}{2}(B+A) = L \cos 83^\circ\ 2'\ 5\tfrac{1}{2}'' = 9{\cdot}0837368.$$

$$L \cos \tfrac{1}{2}(B-A) = L \cos 42^\circ\ 51'\ 35\tfrac{1}{2}'' = 9{\cdot}8651156.$$

$$\tfrac{1}{2}c = 13^\circ\ 41'\ 37''.$$

***Ans.*—**

$B = 54^\circ\ 6'\ 19''$, $C = 116^\circ\ 55'\ 26''$, $c = 90^\circ\ 31'\ 46''$.

$B = 125^\circ\ 53'\ 41''$, $C = 24^\circ\ 12'\ 53\tfrac{1}{3}''$, $c = 27^\circ\ 23'\ 14''$.

69. CASE VI.—*Having given two angles and a side opposite one of them* (A, B, a).

This Case reduces, by aid of the polar triangle, to the preceding, and the direct solution may therefore be obtained by means of formulæ supplemental to those employed in Art. 68, viz.,

$$\sin b = \frac{\sin B}{\sin A} \sin a,$$

$$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A+B),$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b).$$

The same ambiguities will thus present themselves.

Examples.

(1). $a = 59° 28' 27''$, $A = 66° 7' 20''$, $B = 52° 50' 20''$.

$$b = 48° 39' 16'', \text{ or } 131° 20' 44''.$$

Firstly—take the value of b less than a right angle.

$$L \cos \tfrac{1}{2}(a-b) = L \cos 5° 24' 35\tfrac{1}{2}'' = 9{\cdot}9980612.$$
$$L \cos \tfrac{1}{2}(a+b) = L \cos 54° 3' 51\tfrac{1}{2}'' = 9{\cdot}7685470.$$
$$L \cot \tfrac{1}{2}(A+B) = L \cot 59° 28' 50'' = 9{\cdot}7704854.$$

$$\tfrac{1}{2} C = 45°.$$

$$L \cos \tfrac{1}{2}(A+B) = L \cos 59° 28' 50'' = 9{\cdot}7057190.$$
$$L \cos \tfrac{1}{2}(A-B) = L \cos 6° 38' 30'' = 9{\cdot}9970756.$$
$$L \tan \tfrac{1}{2}(a+b) = L \tan 54° 3' 51\tfrac{1}{2}'' = 10{\cdot}1397643.$$

$$\tfrac{1}{2} c = 35° 11' 50\tfrac{3}{4}''.$$

The second value of b is inadmissible (see foot note, Art. 67, Ex. 4), and therefore there is only one solution.

Ans.—

$$b = 48^\circ\ 39'\ 16'', \quad C = 90^\circ, \quad c = 70^\circ\ 23''\ 41\tfrac{1}{2}''.$$

(2). $a = 53^\circ\ 18'\ 20''$, $A = 79^\circ\ 30'\ 45''$, $B = 46^\circ\ 15'\ 15''$.

$$b = 36^\circ\ 5'\ 34\tfrac{3}{4}'', \text{ or } 143^\circ\ 54'\ 25\tfrac{1}{4}''.$$

Firstly—

$$\begin{aligned}
L\cos\tfrac{1}{2}(a-b) &= L\cos\ 8^\circ\ 36'\ 22\tfrac{1}{2}'' = 9{\cdot}9950822.\\
L\cos\tfrac{1}{2}(a+b) &= L\cos 44^\circ\ 41'\ 57\tfrac{1}{4}'' = 9{\cdot}8517528.\\
L\cot\tfrac{1}{2}(A+B) &= L\cot 62^\circ\ 53'\ \ 0'' = 9{\cdot}7093488.
\end{aligned}$$

$$\tfrac{1}{2}C = 35^\circ\ 27'\ 47\tfrac{1}{2}''.$$

$$\begin{aligned}
L\cos\tfrac{1}{2}(A+B) &= L\cos 62^\circ\ 53'\ \ 0'' = 9{\cdot}6587780.\\
L\cos\tfrac{1}{2}(A-B) &= L\cos 16^\circ\ 37'\ 45'' = 9{\cdot}9814457.\\
L\tan\tfrac{1}{2}(a+b) &= L\tan 44^\circ\ 41'\ 57\tfrac{1}{4}'' = 9{\cdot}9954404.
\end{aligned}$$

$$\tfrac{1}{2}c = 25^\circ\ 12'\ 28\tfrac{1}{2}''.$$

[No ambiguity, as in Ex. 1.]

Ans.—

$$b = 36^\circ\ 5'\ 34\tfrac{3}{4}'' \quad C = 70^\circ\ 55'\ 35'', \quad c = 50^\circ\ 24'\ 57''.$$

(3). $a = 46^\circ\ 45'\ 30''$, $A = 73^\circ\ 11'\ 18''$, $B = 61^\circ\ 18'\ 12''$;

therefore $b = 41^\circ\ 52'\ 34\tfrac{3}{4}''$, or $138^\circ\ 7'\ 25\tfrac{1}{4}''$.

Firstly—

$$\begin{aligned}
L\cos\tfrac{1}{2}(a-b) &= L\cos\ 2^\circ\ 26'\ 23\tfrac{1}{4}'' = 9{\cdot}9996061.\\
L\cos\tfrac{1}{2}(a+b) &= L\cos 44^\circ\ 19'\ \ 6\tfrac{3}{4}'' = 9{\cdot}8545894.\\
L\cot\tfrac{1}{2}(A+B) &= L\cot 67^\circ\ 14'\ 45'' = 9{\cdot}6226494.
\end{aligned}$$

$$\tfrac{1}{2}C = 30^\circ\ 21'\ 23\tfrac{1}{4}''.$$

$$\begin{aligned}
L\cos\tfrac{1}{2}(A+B) &= L\cos 67^\circ\ 14'\ 45'' = 9{\cdot}5874618.\\
L\cos\tfrac{1}{2}(A-B) &= L\cos\ 5^\circ\ 56'\ 33'' = 9{\cdot}9976600.\\
L\tan\tfrac{1}{2}(a+b) &= L\tan 44^\circ\ 19'\ \ 6\tfrac{3}{4}'' = 9{\cdot}9896683.
\end{aligned}$$

$$\tfrac{1}{2}c = 20^\circ\ 47'\ 32''.$$

[Only one solution.]

Ans.—

$$b = 41^\circ\ 52'\ 34\tfrac{3}{4}'', \quad C = 60^\circ\ 42'\ 46\tfrac{1}{2}'', \quad c = 41^\circ\ 35'\ 4''.$$

(4). $a = 42^\circ\ 15'\ 20''$, $A = 46^\circ\ 30'\ 40''$, $B = 36^\circ\ 20'\ 20''$.

$b = 33^\circ\ 18'\ 47\tfrac{1}{2}''$, or $146^\circ\ 41'\ 12\tfrac{1}{2}''$.

Firstly—

$L \cos \tfrac{1}{2}(a - b) = L \cos\ 4^\circ\ 28'\ 16\tfrac{1}{4}'' = 9{\cdot}9986763.$

$L \cos \tfrac{1}{2}(a + b) = L \cos 37^\circ\ 47'\ 3\tfrac{3}{4}'' = 9{\cdot}8978042.$

$L \cot \tfrac{1}{2}(A + B) = L \cot 41^\circ\ 25'\ 30'' = 10{\cdot}0543373.$

$\tfrac{1}{2}C = 55^\circ\ 1'\ 37\tfrac{1}{3}''.$

$L \cos \tfrac{1}{2}(A + B) = L \cos 41^\circ\ 25'\ 30'' = 9{\cdot}8749585.$

$L \cos \tfrac{1}{2}(A - B) = L \cos\ 5^\circ\ 5'\ 10'' = 9{\cdot}9982866.$

$L \tan \tfrac{1}{2}(a + b) = L \tan 37^\circ\ 47'\ 3\tfrac{3}{4}'' = 9{\cdot}8894377.$

$\tfrac{1}{2}c = 30^\circ\ 16'\ 3''.$

[Only one solution.]

Ans.—

$b = 33^\circ\ 18'\ 47\tfrac{1}{2}''$, $C = 110^\circ\ 3'\ 14\tfrac{2}{3}''$, $c = 60^\circ\ 32'\ 6''$.

(5). $a = 34^\circ\ 30'$, $A = 61^\circ\ 29'\ 30''$, $B = 24^\circ\ 30'\ 30''$.

$b = 15^\circ\ 30'\ 30\tfrac{1}{2}''$, or $164^\circ\ 29'\ 29\tfrac{1}{2}''$.

$L \cos \tfrac{1}{2}(a - b) = L \cos\ 9^\circ\ 29'\ 44\tfrac{3}{4}'' = 9{\cdot}9940081.$

$L \cos \tfrac{1}{2}(a + b) = L \cos 25^\circ\ 0'\ 15\tfrac{1}{4}'' = 9{\cdot}9572609.$

$L \cot \tfrac{1}{2}(A + B) = L \cot 43^\circ\ 0'\ 0'' = 10{\cdot}0303441.$

$\tfrac{1}{2}C = 49^\circ\ 24'\ 29\tfrac{1}{4}''.$

$L \cos \tfrac{1}{2}(A + B) = L \cos 43^\circ\ 0'\ 0'' = 9.8641275.$

$L \cos \tfrac{1}{2}(A - B) = L \cos 18^\circ\ 29'\ 30'' = 9{\cdot}9769777.$

$L \tan \tfrac{1}{2}(a + b) = L \tan 25^\circ\ 0'\ 15\tfrac{1}{4}'' = 9{\cdot}6687563.$

$\tfrac{1}{2}c = 19^\circ\ 46'\ 56''.$

[Only one solution.]

Ans.—

$b = 15^\circ\ 30'\ 30\tfrac{1}{2}''$, $C = 98^\circ\ 48'\ 58\tfrac{1}{2}''$, $c = 39^\circ\ 33'\ 52''$.

(6). $a = 42^\circ\ 15'\ 20''$, $A = 36^\circ\ 20'\ 20''$, $B = 46^\circ\ 30'\ 40''$;

therefore $b = 55^\circ\ 25'\ 2\tfrac{1}{2}''$, or $124^\circ\ 34'\ 57\tfrac{1}{2}''$.

Firstly—take the value of b less than a right angle.

$$\begin{aligned}
L\cos\tfrac12(a-b) &= L\cos\ 6^\circ\ 34'\ 51\tfrac14'' = 9{\cdot}9971289.\\
L\cos\tfrac12(a+b) &= L\cos 48^\circ\ 50'\ 11\tfrac14'' = 9{\cdot}8183648.\\
L\cot\tfrac12(A+B) &= L\cot 41^\circ\ 25'\ 30'' = 10{\cdot}0543373.\\
\tfrac12 C &= 59^\circ\ 41'\ 13\tfrac34''.\\
L\cos\tfrac12(A+B) &= L\cos 41^\circ\ 25'\ 30'' = 9{\cdot}8749585.\\
L\cos\tfrac12(A-B) &= L\cos\ 5^\circ\ 5'\ 10'' = 9{\cdot}9982866.\\
L\tan\tfrac12(a+b) &= L\tan 48^\circ\ 50'\ 11\tfrac14'' = 10{\cdot}0583343.\\
\tfrac12 c &= 40^\circ\ 43'\ 43\tfrac18''.
\end{aligned}$$

Secondly—take the value of b greater than a right angle.

$$\begin{aligned}
L\cos\tfrac12(a-b) &= L\cos 41^\circ\ 9'\ 48\tfrac34'' = 9{\cdot}8766992.\\
L\cos\tfrac12(a+b) &= L\cos 83^\circ\ 25'\ 8\tfrac34'' = 9{\cdot}0592074.\\
\tfrac12 C &= 82^\circ\ 20'\ 57\tfrac12''.\\
L\tan\tfrac12(a+b) &= L\tan 83^\circ\ 25'\ 8\tfrac34'' = 10{\cdot}9379216.\\
\tfrac12 c &= 81^\circ\ 17'\ 13\tfrac12''.
\end{aligned}$$

Ans.—$b = 55^\circ\ 25'\ 2\tfrac12''$, $C = 119^\circ\ 22'\ 27\tfrac12''$, $c = 81^\circ\ 27'\ 26\tfrac14''$;
or $b = 124^\circ\ 34'\ 57\tfrac12''$, $C = 164^\circ\ 41'\ 55''$, $c = 162^\circ\ 34'\ 27''$.

(7). $a = 59^\circ\ 28'\ 27''$. $A = 52^\circ\ 50'\ 20''$, $B = 66^\circ\ 7'\ 20''$;
therefore $b = 81^\circ\ 15'\ 15''$, or $98^\circ\ 44'\ 45''$.

Firstly—take the value of b less than a right angle.

$$\begin{aligned}
L\cos\tfrac12(a-b) &= L\cos 10^\circ\ 53'\ 24'' = 9{\cdot}9921078.\\
L\cos\tfrac12(a+b) &= L\cos 70^\circ\ 21'\ 51'' = 9{\cdot}5263918.\\
L\cot\tfrac12(A+B) &= L\cot 59^\circ\ 28'\ 50'' = 9{\cdot}7704854.\\
\tfrac12 C &= 59^\circ\ 51'\ 54''.\\
L\cos\tfrac12(A+B) &= L\cos 59^\circ\ 28'\ 50'' = 9{\cdot}7057190.\\
L\cos\tfrac12(A-B) &= L\cos\ 6^\circ\ 38'\ 30'' = 9{\cdot}9970756.\\
L\tan\tfrac12(a+b) &= L\tan 70^\circ\ 21'\ 51'' = 10{\cdot}4475887.\\
\tfrac12 c &= 55^\circ\ 5''\ 25\tfrac14''.
\end{aligned}$$

Secondly—take the value of b greater than a right angle.

$$\begin{aligned}
L\cos\tfrac12(a-b) &= L\cos 19^\circ\ 38'\ 9'' = 9{\cdot}9739806.\\
L\cos\tfrac12(a+b) &= L\cos 79^\circ\ 6'\ 36'' = 9{\cdot}2762872.\\
\tfrac12 C &= 71^\circ\ 12'\ 29\tfrac12''.\\
L\tan\tfrac12(a+b) &= L\tan 79^\circ\ 6'\ 36'' = 10{\cdot}7158207.\\
\tfrac12 c &= 69^\circ\ 22'\ 43''.
\end{aligned}$$

Ans.—$b = 81^\circ\ 15'\ 15''$, $C = 119^\circ\ 43'\ 48''$, $c = 110^\circ\ 10'\ 50\tfrac12''$;
or $b = 98^\circ\ 44'\ 45''$, $C = 142^\circ\ 24'\ 59''$, $c = 138^\circ\ 45'\ 26''$

70. Discussion of the Ambiguous Case.—Before entering upon the general discussion of the ambiguous case of spherical triangles, it will be necessary for the student to be familiar with the analogous ambiguity *in plano*, a complete discussion of which is generally given in works on Plane Trigonometry.

The reasoning is somewhat more complex in the case of spherical triangles, as there are nine distinct cases to be considered, in any of which an ambiguity in the construction of the triangle may present itself. It has been seen in the examples of Arts. 68 and 69, that from the given parts two real solutions were obtained, from which we infer that there can be constructed two distinct triangles satisfying the given conditions.

We shall now examine in detail the variety of cases which may arise when '*two sides and the angle opposite one of them are given*,' leaving the supplemental case, '*having given two angles and the side opposite one of them*,' to be reasoned out in an analogous manner.

71. *Let the given parts be denoted by A, a, b.*

The discussion resolves itself thus into three groups, each containing three separate cases:—

Case I.—When $A = \frac{1}{2}\pi$, and b is equal to, less than, or greater than $\frac{1}{2}\pi$;

Case II.—When $A < \frac{1}{2}\pi$, and b is equal to, less than, or greater than $\frac{1}{2}\pi$;

Case III.—When $A > \frac{1}{2}\pi$, and b is equal to, less than, or greater than $\frac{1}{2}\pi$;

the solution in any case being impossible, evanescent, unique, ambiguous, or indeterminate, according to the value of a.

72. Case I.—$A = \frac{1}{2}\pi$.

For the consideration of the three cases in Art. 71, take any great circle AC (fig. 29) equal to the given side b, and at right angles to the great circle ABX.

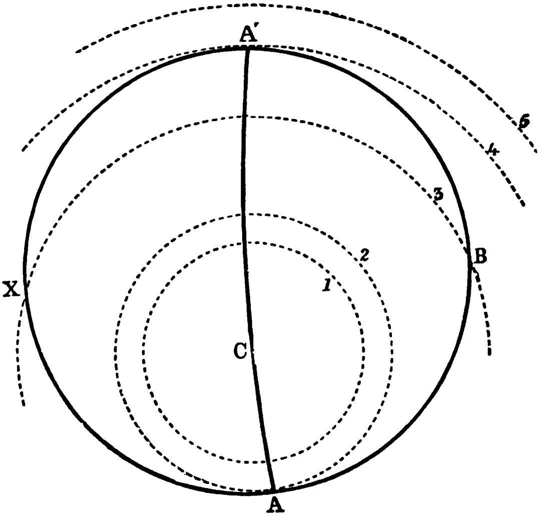

Fig. 29.

Firstly.—Let $b = \frac{1}{2}\pi$; and with C as pole, and a radius equal to a, describe a circle. Now if a be less or greater than a quadrant, this circle being concentric with ABX, the triangle is *impossible*. If a be a quadrant, the circles are coincident, and the triangle is *indeterminate* with the given parts A, a, b, each quadrants.

Secondly.—Let $b < \frac{1}{2}\pi$. With C as pole (fig. 29), and a radius a, describe a circle. Since AC and $A'C$ are unequal, this circle may be situated, relatively to ABX, in five ways—(1) when the circumference falls wholly within ABX; (2) when it touches ABX at A; (3) when it cuts ABX in the points B and X; (4) when it touches ABX at A'; (5) when it falls without ABX. These different positions are drawn on fig. 29, where it will be seen that the points A and A' are diametrically opposite on ABX; also that B and X are equidistant from the points A and A'.

The following results are therefore geometrically manifest:—

If $a < b$,	the triangle is impossible:	see circle	1.
If $a = b$,	„ „ evanescent:	„	2.
If $a > b$ and $< \pi - b$,	two identical solutions:	„	3.
If $a = \pi - b$,	the triangle is evanescent:	„	4.
If $a > \pi - b$,	„ „ impossible:	„	5.

73. *Thirdly.*—Let $b > \frac{1}{2}\pi$. Let A' be the vertex of the triangle to be constructed with the given parts, and let $A'C = b$. With C as pole, describe a circle of radius a, which, as already explained, may assume any of the five positions (fig. 29), according to the magnitude of a. Hence—

If $a < \pi - b$, the triangle is impossible: see circle 1.
If $a = \pi - b$, ,, ,, evanescent: ,, 2.
If $a > \pi - b$ and $< b$, two identical solutions: ,, 3.
If $a = b$, the triangle is evanescent: ,, 4.
If $a > b$, ,, ,, impossible: ,, 5.

74. Case II.—$A < \frac{1}{2}\pi$.

Draw an arc AC (fig. 30) equal to b, and making the given angle A with the great circle $B_1B_2B_3$. . . . . With C

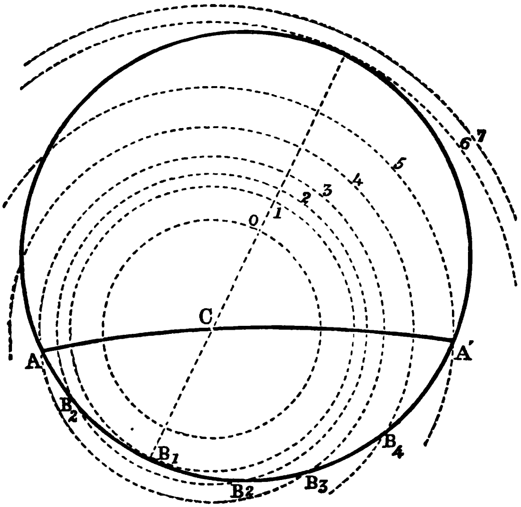

Fig. 30.

as pole, and a radius a, describe a circle. The position of this circle, as in Case I., will depend entirely on the value of the side a; and eight positions, with respect to $B_1B_2B_3$, . . . are shown on fig. 30, viz., circles 0, 1, 2, &c. The circles numbered 0 and 7 respectively have no point in common with $B_1B_2B_3 \ldots$, which we may regard as the base of the triangle under consideration.

We shall discuss the different cases under the heading of this Article—firstly, when $b = \frac{1}{2}\pi$; secondly, when $b < \frac{1}{2}\pi$; thirdly, when $b > \frac{1}{2}\pi$.

Firstly.—Let $b = \frac{1}{2}\pi$. Then C, the vertex of the triangle, is the middle point of the arc AA', and we have—

If $a < A$, the triangle is impossible: see circle 0.
If $a = A$, ,, ,, unique: see circle 1.
If $a > A$ and $< b$, ,, ,, ambiguous: see circle 2.
If $a = b = \frac{1}{2}\pi$,, ,, evanescent: see circles 3 & 5.
If $a > b$, ,, ,, impossible: see circles 6 & 7.

Secondly.—Let $b < \frac{1}{2}\pi$. The following results are evident from fig. 30:—

If $\sin a < \sin b \sin A$, the triangle is impossible.
(See circles 0 and 7.)

If $\sin a = \sin b \sin A$, ,, ,, unique.
(See circle 1.)

If $\sin a > \sin b \sin A$, and $a < b$, ,, ,, ambiguous.
(See circle 2.)

If $a = b$, ,, ,, unique.
(See circle 3.)

If $a > b$, and $< \pi - b$, ,, ,, unique.
(See circle 4.)

If $a = \pi - b$, ,, ,, evanescent.
(See circle 5.)

If $a > \pi - b$, ,, ,, impossible.
(See circles 6 and 7.)

Thirdly.—Let $b > \frac{1}{2}\pi$. Let $A'C$ (fig. 30) be the given side b, and A' the given angle ($= A$).

Proceeding as before, we have—

If $\sin a < \sin b \sin A$, the triangle is impossible.
(See circle 0.)
If $\sin a = \sin b \sin A$, " " unique.
(See circle 1.)
If $\sin a > \sin b \sin A$, and $a < \pi - b$, " " ambiguous.
(See circle 2.)
If $a = \pi - b$, " " unique.
(See circle 3.)
If $a > \pi - b$, and $< b$, " " unique.
(See circle 4.)
If $a = b$, " " evanescent.
(See circle 5.)
If $a > b$, " " impossible.
(See circles 6 and 7.)

75. In the previous Articles, the general cases of ambiguity, when A is equal to, or less than, a quadrant, and a and b any values between 0° and π, have been discussed. The following additional cases are left as exercises for the reader when A is obtuse.

Examples.

1. Given $A > \frac{1}{2}\pi$ and $b < \frac{1}{2}\pi$, the triangle is impossible if $a < b$.

2. In the same case the triangle is unique when a lies between b and $\pi - b$.

3. Given A and b, as in Ex. 1, the triangle is ambiguous when a lies between $\pi - b$ and the greater value of $\sin^{-1}(\sin b \sin A)$.

4. Given A obtuse and b acute, the triangle is impossible if a is greater than the greater value of $\sin^{-1}(\sin b \sin A)$.

5. Given A and b both obtuse, the triangle is impossible when $a < b$.

6. Given A and b both obtuse; find the limiting values of a giving rise to an ambiguity.

Ans. $a > b$, and $< \sin^{-1}(\sin b \sin A)$.

7. What values assigned to the given parts give rise to an *indeterminate* construction?

*8. If a, b, c are each $< \frac{1}{2}\pi$, the greater angle alone *may* exceed $\frac{1}{2}\pi$.

[Let $a > b > c$; then both $\cos b$ and $\cos c > \cos a$, and therefore $> \cos c \cos a$, or $\cos a \cos b$; therefore, &c, Art. 26 (1).]

*9. If a alone $> \frac{1}{2}\pi$, A *must* exceed $\frac{1}{2}\pi$.

[Apply Art. 26 (1).]

*10. If a and b are each $> \frac{1}{2}\pi$, and $c < \frac{1}{2}\pi$; prove that—

(1) the greatest angle (A) must be $> \frac{1}{2}\pi$;
(2) B may be $> \frac{1}{2}\pi$;
(3) C may or may not be $< \frac{1}{2}\pi$.

[Proof as before.]

*11. If $\cos a$, $\cos b$, $\cos c$ are all negative; then $\cos A$, $\cos B$, $\cos C$ are all necessarily negative.

12. In a spherical triangle of the five products,

$\cos a \cos A$, $\cos b \cos B$, $\cos c \cos C$, $\cos a \cos b \cos c - \cos A \cos B \cos C$,

one is negative, the other four being positive.—(COTTERILL.)

[Apply Examples 8, 9, 10, 11.]

76. The Subsidiary Angle.—It has been seen that, when two sides and the included angle of a triangle were given, the value of the third side was made to depend on the values ascertained for the base angles. It is likewise evident, in CASES V. and VI. of this Chapter, that the values of two of the parts depended on the value or values determined for the remaining part. An error in this latter will, consequently, cause an error in the values of parts found from it.

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* *Educational Times*, vol. xli., reprint, p. 89.

By means of the subsidiary angle, each of the parts required is determined from the data in question in a manner independent of one another.

(1) In CASE III., Art. 62, c may be found in terms of a, b, C, from a formula adapted to logarithmic computation: thus—

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

$$= \cos b\,(\cos a + \sin a \tan b \cos C).$$

Let $$\tan\theta = \tan b \cos C. \tag{1}$$

Hence

$$\cos c = \cos b\,(\cos a + \sin a \tan\theta) = \cos b \cos(a-\theta)\sec\theta;$$

or $$\frac{\cos b}{\cos c} = \frac{\cos\theta}{\cos(a-\theta)}. \tag{2}$$

The subsidiary angle θ is found from (1), and c can therefore be calculated from (2).

NOTE.—It is evident, from (2), that θ and $a-\theta$ are the segments of a, made by the perpendicular to it from the opposite angle A. (Cf. Chap. IV., Ex. 11.) The solution of the triangle is thus shown to be equivalent to the solutions of these right-angled triangles.

(2) In CASE IV., Art 67, C may be found directly from

the given parts A, B, c, in a form adapted to logarithmic computation: thus—

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$
$$= \cos B\,(-\cos A + \sin A \tan B \cos c).$$

Let $$\cot\theta = \tan B \cos c.$$

Hence

$$\cos C = \cos B\,(-\cos A + \sin A \cot\theta) = \cos B \sin(A-\theta)\operatorname{cosec}\theta;$$

or $$\frac{\cos B}{\cos C} = \frac{\sin\theta}{\sin(A-\theta)};\text{ therefore, \&c.}$$

It is obvious that the subsidiary angle θ is in this case a segment of the angle A made by the perpendicular from A on the opposite side. (Cf. Chap. IV., Ex. 17.)

Examples.

1. Having given a, b, A, adapt the formula

$$\cot a \sin b = \cot A \sin C + \cos b \cos C$$

to logarithmic computation for the angle C.

Ans. $\tan a : \tan b = \cos\theta : \cos(C-\theta.)$

2. What is the geometrical interpretation of θ in Ex. 1?

[Cf. Art. 44 (2).]

3. Given, as before, a, b, A; show how to find c in a logarithmic form.

Ans. $\cos a : \cos b = \cos(c-\theta) : \cos\theta$, where $\tan\theta = \tan b \cos A$.

[Cf. Art. 26 (1).]

4. Prove the result given in Ex. 3 geometrically.

5. Having given A, B, a; construct geometrically the subsidiary angles required—(1) to calculate C; (2) to calculate c.

Ans. Draw a perpendicular from C to c, and take—(1) the segment of C adjacent to a; (2) the segment of c adjacent to a.

A TREATISE

ON

SPHERICAL TRIGONOMETRY,

WITH APPLICATIONS TO

SPHERICAL GEOMETRY,

AND

NUMEROUS EXAMPLES.

Part II.

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PREFACE

TO THE SECOND PART.

AS the subject-matter of the Second Part, consisting largely of Spherical Geometry, differs widely from that of the first, it seems necessary to state the considerations which induced us to introduce it into a work on Spherical Trigonometry.

In the first place, Spherical Geometry is very closely allied to Spherical Trigonometry, and in some places the two subjects so overlap that separation becomes impossible. Again, the former subject can only be approached through the latter, and it therefore forms a fitting sequel to a work on that subject; unless, indeed, it might rank of such importance as to demand separate treatment, and a separate volume for itself. Owing to the comparative neglect into which it has fallen, this latter appears to us to be out of the question at present; and hoping to rescue it, to some extent, we have taken such notice of it as appears in the later chapters of the present treatise. We have been further urged to this by the absence, as far as we are aware, of any book entering into the subject, or even broaching upon it, except the late Professor Mulcahy's elegant treatise on Modern Geometry, where some of the more elementary principles are introduced. To this work we wish to acknowledge our obligations in so far as that from it we drew some of our first ideas on the subject.

For the sake of compression we have confined ourselves to giving in the text only such theorems as we considered of fundamental importance, and we have placed as examples many elegant and important propositions, analogues of well-known results in Plane Geometry. In most cases the proofs will be found identical with the recognised methods used to prove the corresponding theorems *in plano;* but in many instances the proofs, as well as the results, are original, yet we trust we have in no case introduced originality, except with the object of gaining simplicity. The results are in some places so closely allied to those of Plane Geometry, that their further development has been left to the student. This is notably so in the case of Anharmonic Section and Spherical Involution.

Dismissing the subject of Spherical Geometry, the other chapters of this part (which belong to Spherical Trigonometry) have been written with the same aim as those of

Part I., viz., that each should be as complete as possible in itself. To this end we have introduced into Chap. VI., which treats of the circles related to a triangle, a section on the angular distances of the pole of the Circumcircle from the poles of the In- and Ex-circles, and also a section embracing the general properties of Dr. Hart's Circle. These sections, however, and such other material throughout the work as appeared too difficult for the beginner, have been placed in small type, and may be judiciously omitted on first reading.

In Chap. VII., which treats of Areas, an attempt has been made to reduce to general headings what has generally appeared in a straggling shape. Some novelty has also been introduced into it, *e. g.* a geometrical representation of the Spherical Excess, and a section on Maximum Areas, as well as an introductory section. Chapter VIII. is devoted entirely to the properties of the Chordal Triangle, and the most meagre perusal will render it obvious why it does not appear earlier in the work. Chapters IX. and X. are the result of an attempt to reduce to general headings some propositions apparently of an outlying and miscellaneous character. They have been considerably developed, and the principles in them are so important as not only to justify their insertion, but also to amply repay any time or trouble spent upon them.

The principles of Spherical and Stereographic Projection are of such importance, that we have deemed it advisable to devote an entire chapter to their development and illustration, and when the student has made himself familiar with them, he will find himself possessed of methods of great power and discovery. With respect to the Stereographic Projection, we are largely indebted to "Des Méthodes en Géométrie," par Paul Serret.

Another method (analytical) of attacking and investigating the properties of Spherical Figures we have used considerably, and we had hoped to introduce a short notice of it into this part, but want of space has compelled us to omit it, at least for the present.

In conclusion, we desire to thank Mr. C. E. M'Vicker for several useful suggestions, and for the interest he exhibited in the proofs of the later chapters. We regret, however, that we have not been able to obtain some other valuable assistance from our mathematical friends.

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August, 1886.

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CHAPTER VI.

ON THE CIRCLES RELATED TO A TRIANGLE.

SECTION I.

The In-Circles and Ex-Circles.

77. FOUR small circles can be described each touching the sides of a given triangle. The existence of these circles has been referred to in Art. 18, and the demonstrations there were seen to be precisely the same as for the corresponding circles *in plano*. When the points of contact with the sides lie between the vertices, the circle is termed the *Inscribed*, or *In-circle;* and its pole is the intersection of the internal bisectors of the angles of the given triangle. When the points of contact are in triads on one side, and on the other two produced (fig. 31), the circles are said to be *escribed* to the triangle; and the poles of the escribed, or *Ex-circles*, are at the intersections of the bisectors of the external angles. It is therefore manifest that the three ex-circles of any triangle are the in-circles of its colunar, or associated triangles.

The angular radii of the in-circle and ex-circles are usually denoted by the letters r, r_1, r_2, r_3, or, r, r_a, r_b, r_c.

In fig. 31 the six bisectors of the angles of the triangle ABC pass in triads through the four points O, O_1, O_2, O_3; these points being respectively the poles of the in-circle and ex-circles.

Moreover, O_1 is the pole of the circle escribed to the side a, and also the pole of the in-circle of the colunar triangle on the same side; and a similar remark applies to the points O_2 and O_3.

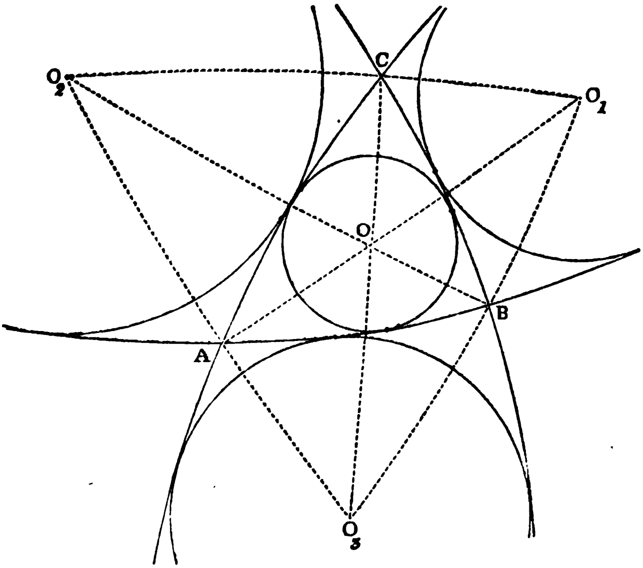

Fig. 31.

Examples.

1. The tangents from A, B, C to the in-circle are $s - a$, $s - b$, $s - c$.
2. From A to the ex-circles, s, $s - c$, $s - b$, &c.
3. To the four circles from any vertex, s, $s - a$, $s - b$, $s - c$.
4. The point O is the orthocentre* of the triangle $O_1O_2O_3$.

---

* Hence, generally, the perpendiculars of a spherical triangle meet in a point (cf. Arts. 21 and 54).

5. The common tangents (direct) to the ex-circles, taken in pairs, are $b+c$, $c+a$, $a+b$.

6. The transverse common tangents to the ex-circles are equal to the sides of the given triangle.

7. The direct common tangents to the in-circle and each of the ex-circles are, respectively, a, b, c; the transverse common tangents are $\frac{1}{2}(b-c)$; $\frac{1}{2}(c-a)$; $\frac{1}{2}(a-b)$.

8. Prove the following :—

$$\cos s + \cos (s-a) + \cos (s-b) + \cos (s-c) = 4 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c.$$

9. $\cos (s-b) + \cos (s-c) - \cos (s-a) - \cos s = 4 \cos \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin \tfrac{1}{2} c.$

10. $\sin (s-a) + \sin (s-b) + \sin (s-c) - \sin s = 4 \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin \tfrac{1}{2} c.$

11. $\sin s + \sin (s-b) + \sin (s-c) - \sin (s-a) = 4 \sin \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c.$

12.
$$\begin{aligned} &\sin (s-b) \sin (s-c) + \sin (s-c) \sin (s-a) + \sin (s-a) \sin (s-b) \\ &\quad + \sin s \sin (s-a) + \sin s \sin (s-b) + \sin s \sin (s-c) \\ &\qquad = \sin b \sin c + \sin c \sin a + \sin a \sin b. \end{aligned}$$

[In Exs. 12 and 13 resolve each term on the left-hand side into a difference of two cosines; therefore, &c.]

13.
$$\begin{aligned} &\sin (s-b) \sin (s-c) + \sin (s-c) \sin (s-a) - \sin (s-a) \sin (s-b) \\ &\quad + \sin s \sin (s-a) + \sin s \sin (s-b) - \sin s \sin (s-c) \\ &\qquad = \sin b \sin c + \sin c \sin a - \sin a \sin b. \end{aligned}$$

14. $\sin^2 s + \sin^2 (s-a) + \sin^2 (s-b) + \sin^2 (s-c) = 2 (1 - \cos a \cos b \cos c).$

[For $2 \sin^2 s = 1 - \cos 2s$, &c.]

15. $\sin^2 s + \sin^2 (s-a) - \sin^2 (s-b) - \sin^2 (s-c) = 2 \cos a \sin b \sin c.$ (Cf. Ex. 9.)

15*a*. $\cos^2 s + \cos^2 (s-a) + \cos^2 (s-b) + \cos^2 (s-c) = 2 (1 + \cos a \cos b \cos c).$
$\cos^2 s + \cos^2 (s-a) - \cos^2 (s-b) - \cos^2 (s-c) = -2 \cos a \sin b \sin c.$

16. $$\cos OO_1 : \cos OO_2 : \cos OO_3 = \frac{\cos r_1}{\cos (s-a)} : \frac{\cos r_2}{\cos (s-b)} : \frac{\cos r_3}{\cos (s-c)}.$$

[For $\cos OO_1 = \cos OC \cos O_1C = \cos r \cos r_1 \cos (s-b) \cos (s-c)$, &c.]

17. The sides of the triangle $O_1O_2O_3$ bisect externally the vertical angles of the colunars of ABC.

18. $\tan O_1O_3 \tan O_1B = \tan O_1A \tan OO_1 = \tan O_1O_2 \tan O_1C.$

(Cf. p. 144, Ex. 3.)

19. $$\frac{\cos O_1}{\sin \frac{1}{2} A} = \frac{\cos O_2}{\sin \frac{1}{2} B} = \frac{\cos O_3}{\sin \frac{1}{2} C} = \cos s.$$

[From the triangle BCO_1, $\cos O_1 + \sin \frac{1}{2} B \sin \frac{1}{2} C = \cos \frac{1}{2} B \cos \frac{1}{2} C \cos a$, &c.]

78. **The In-circle.**—*To find the angular radius of the in-circle of a triangle.*

By the preceding Article, and p. 33 (4), if Z denote the point of contact with the side c (fig. 31), we have

$$\tan r = \tan \tfrac{1}{2} A \sin AZ = \tan \tfrac{1}{2} A \sin (s - a)$$
$$= \tan \tfrac{1}{2} B \sin (s - b) = \tan \tfrac{1}{2} C \sin (s - c). \quad (1)$$

(Art. 44 (4).)

Again, $\tan \frac{1}{2} A \sin (s - a)$ is readily transformed into a symmetric function of the sides by substituting the value of $\tan \frac{1}{2} A$ given in Art. 33 (7).

Hence $$\tan r = \frac{n}{\sin s}. \quad (2)$$

Similarly, by substituting the value of $\sin (s - a)$, given in Art. 33, Ex. 5, we have

$$\tan r = \frac{\sin \frac{1}{2} B \sin \frac{1}{2} C}{\cos \frac{1}{2} A} \sin a. \quad (3)$$

Hence, by Art. 36 (12),

$$\tan r = \frac{N}{2 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C}, \quad (4)$$

an equation which is equivalent to the following:—

$$\cot r = \frac{1}{2N} [\cos S + \cos (S - A) + \cos (S - B) + \cos (S - C)]. \quad (5)$$

(Art. 77, Ex. 8.)

79. **The Ex-circles.**—*To find the angular radii of the ex-circles of a triangle.*

Since the circle escribed to the side a is the in-circle of the colunar triangle, the parts of which are a, $\pi - b$, $\pi - c$, A, $\pi - B$, $\pi - C$, the problem becomes identical with that of Art. 78; and the value of r_a is got by substituting for b, c, B, C, their supplements in the five equations given in that Article.

We thus arrive at the following results:—

$$\tan r_a = \tan \tfrac{1}{2} A \sin s \qquad (1)$$

$$= \frac{n}{\sin (s - a)} \qquad (2)$$

$$= \frac{\cos \frac{1}{2} B \cos \frac{1}{2} C}{\cos \frac{1}{2} A} \sin a \qquad (3)$$

$$= \frac{N}{2 \cos \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C} \qquad (4)$$

$$\cot r_a = \frac{1}{2N} [- \cos S - \cos (S - A) + \cos (S - B) + \cos (S - C)]. \qquad (5)$$

(See Art. 77, Ex. 9.)

The student may also verify these equations by methods similar to those employed in Art. 78 for the in-circle.

80. The colunar triangles on the sides b and c have each two parts, b and B, c and C, equal to parts of the primitive, while their remaining parts are the supplements in the former case of a, c, A, C, and in the latter, of a, b, A, B.

The values of the radii r_b and r_c are therefore found in the way shown for r_a; or they may be obtained from the values of r_a by appropriate interchanges of letters.

Thus

$$\tan r_b = \frac{n}{\sin(s-b)}, \text{ \&c.,}$$

and

$$\tan r_c = \frac{n}{\sin(s-c)}, \text{ \&c.}$$

Examples.

1. Prove the following:—

$$\cot r_1 : \cot r_2 : \cot r_3 : \cot r = \sin(s-a) : \sin(s-b) : \sin(s-c) : \sin s.$$

2. $\tan r \tan r_1 \tan r_2 \tan r_3 = n^2.$

$\cot r \tan r_1 \tan r_2 \tan r_3 = \sin^2 s.$

$\tan r \cot r_1 \tan r_2 \tan r_3 = \sin^2(s-a).$

$\tan r \tan r_1 \cot r_2 \tan r_3 = \sin^2(s-b).$

$\tan r \tan r_1 \tan r_2 \cot r_3 = \sin^2(s-c).$

3. $\cot r \sin s = \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C.$

4. $$\tan r_1 + \tan r_2 + \tan r_3 - \tan r = \frac{4N \sin S}{\sin A \sin B \sin C}.$$

[For $\Sigma(\tan r_1) - \tan r = \dfrac{4N}{\sin A \sin B \sin C}$, multiplied by

$(\sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C + \,.\,.\, + \,.\,.\,.\, - \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C)$].

5. $$\cot r_1 + \cot r_2 + \cot r_3 - \cot r = -\frac{2\cos S}{N} = \frac{4 \sin \tfrac{1}{2}a \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c}{n}.$$

(Cf. Art. 77, Ex. 10.)

6. $$\tan r_1 : \tan r_2 : \tan r_3 = \frac{\sin a}{1+\cos A} : \frac{\sin b}{1+\cos B} : \frac{\sin c}{1+\cos C}.$$

7. $$\frac{\tan r_1 + \tan r_2 + \tan r_3 - \tan r}{\cot r_1 + \cot r_2 + \cot r_3 - \cot r} = -\frac{2N^2 \tan S}{\sin A \sin B \sin C}$$

$$= \tfrac{1}{2}(1 + \cos a + \cos b + \cos c).$$

[By Exs. 4 and 5. See also the value of $\tan S$ on combining the results in p. 67, Ex. 4, and p. 74, Ex. 28.]

8. What are the corresponding theorems *in plano* of Exs. 4 and 5?

Ans. $r_1 + r_2 + r_3 - r = 4R.$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r} = 0.$$

9. $$\cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3 + \cot^2 r = \frac{2\,(1 - \cos a \cos b \cos c)}{n^2}.$$

10. $$\frac{1}{\sin^2 r} + \frac{1}{\sin^2 r_1} - \frac{1}{\sin^2 r_2} - \frac{1}{\sin^2 r_3} = \frac{2 \cos a \sin b \sin c}{n^2}.$$

(Cf. Art. 77, Exs. 14 and 15.)

11. $$\cot r_2 \cot r_3 + \cot r_3 \cot r_1 \pm \cot r_1 \cot r_2 + \cot r\,(\cot r_1 + \cot r_2 \pm \cot r_3)$$
$$= \frac{\sin b \sin c + \sin c \sin a \pm \sin a \sin b}{n^2}.$$

(Art. 77, Exs. 12 and 13.)

12. $$\tan r_2 \tan r_3 + \tan r_3 \tan r_1 \pm \tan r_1 \tan r_2 + \tan r\,(\tan r_1 + \tan r_2 \pm \tan r_3)$$
$$= \sin b \sin c + \sin c \sin a \pm \sin a \sin b.$$

[Since $\tan r_1 \tan r_2 = \sin s \sin (s - c)$, and $\tan r \tan r_1 = \sin (s - b) \sin (s - c)$, therefore, &c.]

SECTION II.

The Circumcircles.

81. **The Circumcircle.**—The small circle passing through the vertices of a spherical triangle is called the *circumscribing* or *circumcircle* of the triangle. It is likewise the circumcircle of the given chordal triangle, and its pole is the intersection with the sphere of the line joining the centres of the sphere and circle.

By reasoning analogous to Euc. IV. v., the pole of the circumcircle may be obtained by bisecting the sides at right angles by arcs of great circles. These arcs concur at a point H (fig. 32), the pole of the circumcircle of the triangle ABC [Art. 19 (10)], and $AH = BH = CH = R$, the angular radius of the circle.

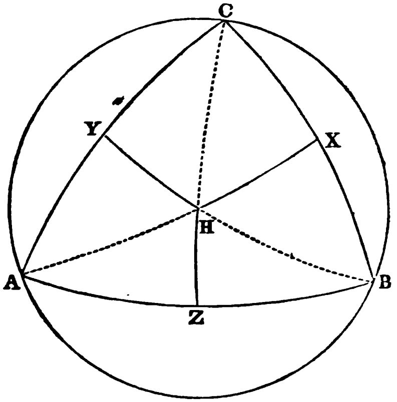

Fig. 32.

82. If a point be equidistant from three given great circles, it is obviously equidistant from the poles of those circles; and if r denote the common angular distance in either case, $90 - r$ is the common angular distance in the other.

In the former case the point may be regarded as one of the four poles of the in-circles of the triangle and its three colunars formed by the great circles; in the latter it is one of the poles of the four circumcircles to the polar triangle and its three colunars. Hence—

1°. *The pole of the in-circle of any triangle is the pole of the circumcircle to the polar triangle.*

2°. *The radius of the in-circle of a triangle is the complement of the radius of the circumcircle to the polar triangle.*

And, more generally—

The radii of the in-circle and three ex-circles of a triangle are the complements of the circumradii of the polar and its associated triangles.

Hence it follows, that if the radius of the in-circle be given in terms of certain known parts (e.g. a, b, c) of a triangle, the radius of the circumcircle of the polar triangle (and therefore of *any triangle*) is also given as a function of the corresponding parts (e.g. A, B, C) of that triangle.

Thus the equation of Art. 78 (2),

$$\tan r = \frac{n}{\sin s},$$

transforms directly into

$$\cot R = \frac{N}{\sin\left(\frac{\pi - A + \pi - B + \pi - C}{2}\right)}$$

for the polar triangle. Hence we have generally for any triangle

$$\tan R = -\frac{\cos S}{N}.$$

Similarly, the remaining values for tan r given in Art. 78 may be transformed into analogous results, which hold generally for the radius of the circumcircle of any triangle.

Also, from the values given in Art. 79 for the angular radii of the ex-circles may be deduced corresponding values for the circumradii of the associated triangles.

On account of their great importance, and the frequent use made afterwards of the different forms of the value of tan R, they are made the subject of discussion in the next Article.

Examples on Fig. 32.

1. Prove that the angles BCH, CAH, and ABH, are respectively equal to $S - A$, $S - B$, and $S - C$.

2. Having given of a triangle the base c and $S - C$, find the locus of the vertex.

[The pole of the circumcircle is fixed; therefore, &c.]

Analogue *in plano* is, "given the base and vertical angle," &c.

3. Having given one side and the sum of the angles; find the locus of the vertex of the colunar triangle on the given side.

[Let $2S'$ denote the sum of the angles of the colunar triangle; then, since $S + S' = \pi + C$, the problem reduces to that of Ex. 2].

4. The point O (fig. 10) is the pole of the circumcircle of the colunar triangle on the base AB.

5. The triangle formed by joining the middle points of the sides is supplemental to the one whose vertices are the poles of the circumcircles of the associated triangles.

6. If S, S_1, S_2, S_3 denote the sums of the angles of a triangle and its three colunars; prove that

$$S + S_1 + S_2 + S_3 = 3\pi.$$

7. The circumcircles of the colunar triangles intersect the circumcircle of the primitive at angles supplemental to those of the given triangle.

8. Prove that if H_1, H_2, H_3 denote the poles of the circumcircles of the colunar triangles,

$$\tan HH_1 \sin \tfrac{1}{2} a = \frac{\sin A}{\sin (S - B) \sin (S - C)},$$

with similar expressions for tan HH_2 and tan HH_3.

[For if X be the middle point of the side a,

$$\tan HX = \tan (S - A) \sin \tfrac{1}{2} a, \text{ and } \tan H_1X = - \tan S \sin \tfrac{1}{2} a.$$

Hence $\tan HH_1 = \sin\frac{1}{2}a \dfrac{\tan(S-A)-\tan S}{1+\sin^2\frac{1}{2}a\tan S\tan(S-A)} = \&c.,$ by Art. 36].

9. $\tan HH_1 : \tan HH_2 : \tan HH_3$

$$= \cos\tfrac{1}{2}a\sin(S-A) : \cos\tfrac{1}{2}b\sin(S-B) : \cos\tfrac{1}{2}c\sin(S-C).$$

10. The angles subtended at H by the sides of the triangle $H_1H_2H_3$ are the supplements of the angles of the chordal triangle.

[For the angles BHC, CHA, AHB are double the angles of the chordal triangle. See Chap. VIII.].

83. *To find the angular radius of the circumcircle of a triangle.*

From fig. 32,

$$\frac{\tan AZ}{\tan AH} = \cos(S-C).$$

Hence

$$\tan R = \frac{\tan\frac{1}{2}a}{\cos(S-A)} = \frac{\tan\frac{1}{2}b}{\cos(S-B)} = \frac{\tan\frac{1}{2}c}{\cos(S-C)}. \tag{1}$$

(Cf. p. 76, Ex. 35.)

Again,

$$\frac{\tan\frac{1}{2}a}{\cos(S-A)}$$

may be transformed into a symmetric function of the angles, by substituting for $\tan\frac{1}{2}a$ its value given in Art. 36 (11); therefore

$$\tan R = -\frac{\cos S}{N}. \tag{2}$$

(Cf. Art. 82.)

Now,

$$\frac{\cos(S-A)}{\cos S} = -\cot\tfrac{1}{2}b\cot\tfrac{1}{2}c.$$

(Art. 36, Ex. 6.)

Hence, by substituting in (1) this value of $\cos(S-A)$, we get

$$\tan R = -\frac{\tan\frac{1}{2}a\tan\frac{1}{2}b\tan\frac{1}{2}c}{\cos S}. \tag{3}$$

(Cf. p. 76, Ex. 35.)

Also (by p. 75, Ex. 31),

$$\cos(S-A) = \frac{n}{2\cos\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c};$$

therefore, from (1),

$$\tan R = \frac{2\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c}{n}; \tag{4}$$

or, by Art. 77, Ex. 10,

$$\tan R = \frac{1}{2n}\left[\sin(s-a)+\sin(s-b)+\sin(s-c)-\sin s\right]. \tag{5}$$

84. **Circumcircle of Colunar Triangles.** — *To find the angular radii of the circumcircles of the three colunar triangles.*

Let R_1, R_2, R_3 be the angular radii of the circumcircles of the colunar triangles on the sides a, b, c, respectively; then, since R_1 is the circumradius of a triangle of parts a, $\pi-b$, $\pi-c$; A, $\pi-B$, $\pi-C$, we have, from Art. 83,

$$\tan R_1 = -\frac{\tan\frac{1}{2}a}{\cos S}. \tag{1}$$

$$\tan R_1 = \frac{\cos(S-A)}{N}. \tag{2}$$

$$\tan R_1 = \frac{\tan\frac{1}{2}a\cot\frac{1}{2}b\cot\frac{1}{2}c}{\cos(S-A)}. \tag{3}$$

$$\tan R_1 = \frac{2\sin\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}{n}. \tag{4}$$

$$\tan R_1 = \frac{1}{2n}\left[\sin s - \sin (s-a) + \sin (s-b) + \sin (s-c)\right]. \quad (5)$$

(See Art. 77, Ex. 11.)

Also,

$$\tan R_2 = -\frac{\tan \frac{1}{2} b}{\cos S} = \frac{\cos (S-B)}{N} = \&c.,$$

and

$$\tan R_3 = -\frac{\tan \frac{1}{2} c}{\cos S} = \frac{\cos (S-C)}{N} = \&c.$$

Examples.

1. Prove the following :—

$\tan R_1 : \tan R_2 : \tan R_3 = \cos (S-A) : \cos (S-B) : \cos (S-C)$.

2. $\cot R \cot R_1 \cot R_2 \cot R_3 = N^2$.

$\tan R \cot R_1 \cot R_2 \cot R_3 = \cos^2 S$.

$\cot R \tan R_1 \cot R_2 \cot R_3 = \cos^2 (S-A)$,

&c., &c.

3. (α) $\tan R_1 + \tan R_2 = \cot r + \cot r_3$.

(β) $\tan R_1 + \tan R_2 + \tan R_3 - \tan R = 2 \cot r$.

4. $\tan R + \cot r = \tan R_1 + \cot r_1 = \&c.$

$= \frac{1}{2}(\cot r + \cot r_1 + \cot r_2 + \cot r_3)$.

5. (α) $\tan^2 R + \tan^2 R_1 + \tan^2 R_2 + \tan^2 R_3 = \dfrac{2(1+\cos A \cos B \cos C)}{N^2}$.

(β) $\cot^2 r + \cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3 = \dfrac{2(1-\cos a \cos b \cos c)}{n^2}$.

(Art. 77, Ex. 15 *a*.)

(γ) $\dfrac{\tan^2 R + \tan^2 R_1 + \tan^2 R_2 + \tan^2 R_3}{\cot^2 r + \cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3} = 1$. (Art. 36, Ex. 7.)

6. (α) $\tan^2 R + \tan^2 R_1 - \tan^2 R_2 - \tan^2 R_3 = -\dfrac{2\cos A \sin B \sin C}{N^2}$.

(β) $\cot^2 r + \cot^2 r_1 - \cot^2 r_2 - \cot^2 r_3 = \dfrac{2\cos a \sin b \sin c}{n^2}$.

(γ) $\dfrac{\tan^2 R + \tan^2 R_1 - \tan^2 R_2 - \tan^2 R_3}{\cot^2 r + \cot^2 r_1 - \cot^2 r_2 - \cot^2 r_3} = -\dfrac{\cos A}{\cos a}$.

(*Vide* Art. 77, Ex. 15 *a*, and Art. 36, Ex. 2.)

7. $$\frac{\tan R}{\tan R_1} = \tan \tfrac{1}{2} b \tan \tfrac{1}{2} c. \qquad \text{(Cf. Art. 36, Ex. 6.)}$$

8. In an equilateral triangle, $\tan R = 2 \tan r$.

9. Construct an equilateral triangle in a given small circle.

[By Art. 83 (1), $\cot \frac{1}{2} A = \sqrt{3} \cos R$; therefore, &c. Or thus:—

$$\sin \tfrac{1}{2} a = \sin R \sin 60^\circ = \frac{\sqrt{3}}{2} \sin R.]$$

10. Construct an equilateral triangle about a given small circle.

[From Ex. 9, or otherwise, we have at once

$$\tan \tfrac{1}{2} a = \sqrt{3} \sin r, \text{ \&c.}]$$

Section III.

Angular distances of the pole of the circumcircle from the poles of the in-circle and ex-circles of a triangle.

85. *Lemma.*—It will be necessary, before proceeding to the discussion of the problems in this section, to prove two propositions which we shall have occasion to make use of hereafter.

$$(1).\ (\cot r + \tan R)^2 = \frac{1}{4n^2} (\sin a + \sin b + \sin c)^2 - 1.$$

$$(2).\ (\cot r_1 - \tan R)^2 = \frac{1}{4n^2} (\sin b + \sin c - \sin a)^2 - 1.$$

To prove (1):—By Art. 78 (2), and Art. 83 (4),

$$\cot r + \tan R = \frac{\sin \frac{1}{2}(a+b+c) + 2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{n}$$

$$= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c + \sin \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c + \ldots + \ldots}{n}$$

$$= \frac{\sin \frac{1}{2} a \cos \frac{1}{2}(b-c) + \cos \frac{1}{2} a \sin \frac{1}{2}(b+c)}{n}.$$

Hence, on squaring and multiplying numerator and denominator of the right-hand side of the equation by 4, we get

$$(\cot r + \tan R)^2 =$$
$$\frac{(1-\cos a)(1+\cos\overline{b-c}) + 2\sin a\,(\sin b + \sin c) + (1+\cos a)(1-\cos\overline{b+c})}{4n^2},$$

which on reduction becomes, by the aid of Art. 30,

$$(\cot r + \tan R)^2 + 1 = \frac{1}{4n^2}(\sin a + \sin b + \sin c)^2;$$

or, since $$\frac{\sin a}{n} = \frac{\sin A}{N}, \qquad \text{(Art. 36, Ex. 2.)}$$

$$(\cot r + \tan R)^2 + 1 = \frac{1}{4N^2}(\sin A + \sin B + \sin C)^2.$$

To prove (2):—By Art. 79 (2) and Art. 83 (4),

$$\cot r_1 - \tan R = \frac{\sin\frac{1}{2}(b+c-a) - 2\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c}{n}$$
$$= \text{etc.,} \quad \text{etc.,} \quad \text{as in (1).}$$

Similarly, $$(\cot r_2 - \tan R)^2 = \frac{1}{4n^2}(\sin c + \sin a - \sin b)^2 - 1,$$

and $$(\cot r_3 - \tan R)^2 = \frac{1}{4n^2}(\sin a + \sin b - \sin c)^2 - 1.$$

Exercise.

Prove the relation—

$$\sqrt{1 + (\cot r_1 - \tan R)^2} + \sqrt{1 + (\cot r_2 - \tan R)^2} + \sqrt{1 + (\cot r_3 - \tan R)^2}$$
$$= \sqrt{1 + (\cot r + \tan R)^2}.$$

86. *Expression for the angular distance between the poles of the circumcircle and in-circle in terms of the radii of the circles.*

Let H and O be the poles (fig. 33) of the circumcircle and in-circles of the triangle ABC; OZ and HZ' secondaries to the side AB.

Now $AZ = s - a$ and $AZ' = \frac{1}{2}c$; therefore $ZZ' = \frac{1}{2}(a - b)$. Let δ denote the angular distance HO; then (p. 103, Ex. 14),

$$\cos\delta = \sin HZ' \sin OZ + \cos HZ' \cos OZ \cos\tfrac{1}{2}(a - b).$$

Also,

$$OZ = r;\ \cos HZ' = \frac{\cos R}{\cos\frac{1}{2}c},\ \text{and}\ \sin HZ' = \sin R \sin(S - C).$$

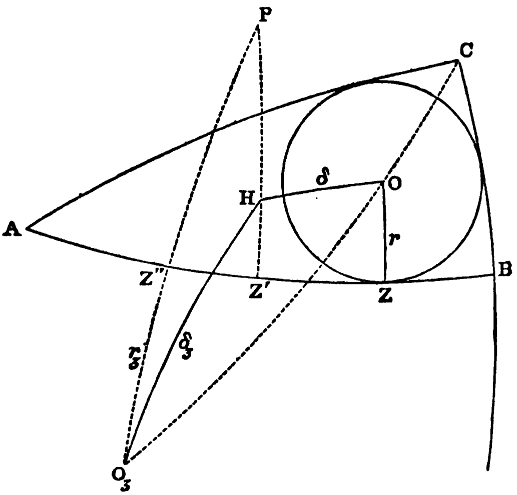

Fig. 33.

Therefore, by substitution,

$$\cos\delta = \sin R \sin r \sin(S - C) + \cos R \cos r\,\frac{\cos\frac{1}{2}(a - b)}{\cos\frac{1}{2}c};$$

or, by Delambre's Analogies, p. 98 (3),

$$\cos\delta = \sin R \sin r \sin(S - C) + \cos R \cos r\,\frac{\sin\frac{1}{2}(A + B)}{\cos\frac{1}{2}C}$$

$$= \cos R \sin r \left\{\frac{-\cos S \sin(S - C)}{N} + \frac{2\cos\frac{1}{2}A \cos\frac{1}{2}B \sin\frac{1}{2}(A + B)}{N}\right\},$$

[Art. 83 (2) and Art. 78 (4).]

which reduces to

$$\cos\delta = \cos R \sin r \left(\frac{\sin A + \sin B + \sin C}{2N}\right).$$

Hence, by Art. 85 (1),

$$\cos^2\delta = \cos^2 R\sin^2 r\,\{(\cot r + \tan R)^2 + 1\}$$
$$= \cos^2 R\sin^2 r + \cos^2(R - r).^*$$

87. *Expressions for the angular distances between the poles of the circum- and excircles in terms of the radii of the circles.*

Let O_3 be the pole of the excircle to the side c (fig. 33), and let $HO_3 = \delta_3$. The secondaries, HZ' and O_3Z'', will meet at P the pole of AB; and since $AZ'' = s - b$, $Z'Z'' = \frac{1}{2}(a - b)$; then, in the triangle PHO_3,

$$\cos\delta_3 = \cos(90 + r_3)\sin HZ' + \sin(90 + r_3)\cos HZ'\cos\tfrac{1}{2}(a - b);$$

or, by substitutions similar to those made in the preceding Article,

$$\cos\delta_3 = -\sin r_3\sin R\sin(S - C) + \cos r_3\cos R\,\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}$$

$$= \cos R\sin r_3\left\{\frac{\cos S\sin(S - C)}{N} + \frac{2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}(A+B)}{N}\right\}.$$

Therefore, by reduction, we get

$$\cos\delta_3 = \frac{\cos R\sin r_3}{2N}(\sin A + \sin B - \sin C);$$

or, by Art. 85 (2),

$$\cos^2\delta_3 = \cos^2 R\sin^2 r_3\,\{(\cot r_3 - \tan R)^2 + 1\}$$
$$= \cos^2 R\sin^2 r_3 + \cos^2(R + r_3).$$

Similarly, $\qquad \cos^2\delta_2 = \cos^2 R\sin^2 r_2 + \cos^2(R + r_2),$

and $\qquad \cos^2\delta_1 = \cos^2 R\sin^2 r_1 + \cos^2(R + r_1).$

---

* This relation (since it is independent of the parts of the triangle) affirms that when two circles are so related that a triangle can be inscribed in one and circumscribed to the other, an infinite number of such triangles can be constructed. It may therefore be easily deduced from the particular case in which one side of the triangle passes through the pole of the circumcircle.

Examples.

1. Prove that

(1) $\sin^2 \delta = \sin^2 (R - r) - \cos^2 R \sin^2 r.$

(2) $\sin^2 \delta_1 = \sin^2 (R + r_1) - \cos^2 R \sin^2 r_1.$

(3) $\sin^2 \delta_2 = \sin^2 (R + r_2) - \cos^2 R \sin^2 r_2.$

(4) $\sin^2 \delta_3 = \sin^2 (R + r_3) - \cos^2 R \sin^2 r_3.$

2. What theorems *in plano* are analogous to those of Arts. 86 and 87?

Ans. $D^2 = R^2 - 2Rr.$

$D_1{}^2 = R^2 + 2Rr_1$, etc., etc.

3. Prove the formula $$\frac{\cos \delta}{\sin r \sin R} = \frac{\sin a + \sin b + \sin c}{4 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}.$$

SECTION IV.

Dr. Hart's Circle.*

88. On the sphere, as on a plane surface, a circle may be described touch-

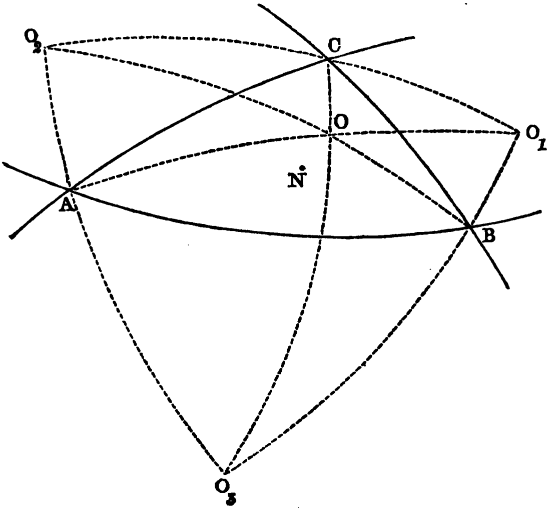

Fig. 34.

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* Now Sir A. S. Hart, Vice-Provost, Trinity College, Dublin.

ing the inscribed and three escribed circles of a triangle. The spherical properties of this circle have been described by Dr. Hart and Dr. Salmon in the *Quarterly Journal of Mathematics*, vol. vi., p. 67: an independent proof of its existence and a few of its properties are here given.

Let N (fig. 34) be the pole of the supposed circle of radius ρ; then the arc

$$NO = \rho - r, \quad NO_1 = \rho + r_1, \quad NO_2 = \rho + r_2, \quad NO_3 = \rho + r_3.$$

Now since in the triangle NO_2O_3 a point A is taken in the base,

$$\cos(\rho + r_2)\sin AO_3 + \cos(\rho + r_3)\sin AO_2 = \cos AN \sin O_2O_3$$

(by Art. 40.)

Similarly for the triangle ANO_1, and the point O in the base,

$$\cos AN \sin OO_1 + \cos(\rho + r_1)\sin AO = \cos(\rho - r)\sin AO_1.$$

Hence, by eliminating $\cos AN$ from these equations,

$$\cos(\rho + r_1)\sin AO \sin O_2O_3 + \cos(\rho + r_2)\sin AO_3 \sin OO_1$$
$$+ \cos(\rho + r_3)\sin AO_2 \sin OO_1 = \cos(\rho - r)\sin AO_1 \sin O_2O_3.$$

This equation transforms directly into

$$\cos(\rho + r_1)\frac{\sin AO \sin O_2O_3}{\sin AO_1 \sin O_2O_3} + \cos(\rho + r_2)\frac{\sin BO \sin O_3O_1}{\sin BO_2 \sin O_3O_1}$$
$$+ \cos(\rho + r_3)\frac{\sin CO \sin O_1O_2}{\sin CO_3 \sin O_1O_2} = \cos(\rho - r),$$

by dividing it by $2n$ of the triangle $O_1 O_2 O_3$, and remembering that

$$\sin AO_2 : \sin O_1 O_2 :: \sin CO : \sin OO_1, \text{ etc., etc.}$$

Hence

$$\frac{\cos(\rho + r_1)\sin r}{\sin r_1} + \frac{\cos(\rho + r_2)\sin r}{\sin r_2} + \frac{\cos(\rho + r_3)\sin r}{\sin r_3} = \cos(\rho - r),$$

or

$$\cos\rho\,(\cot r_1 + \cot r_2 + \cot r_3 - \cot r) = 4\sin\rho;$$

but

$$\cot r_1 + \cot r_2 + \cot r_3 - \cot r = 2\tan R; \quad \text{(Art. 79, Ex. 5.)}$$

therefore

$$2\tan R = 4\tan\rho;$$

or

$$\tan\rho = \tfrac{1}{2}\tan R.$$

This shows the existence of a circle touching the in- and three excircles of a triangle, since we get a real value for the radius ρ.*

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* The above method of showing the existence of a circle touching the in- and excircles of a spherical triangle suggests an analogous deduction of the corresponding theorem *in plano.*

89. **Analogous Theorem in Plane.**—The formula $\tan\rho = \frac{1}{2}\tan R$ becomes on a plane $\rho = \frac{1}{2}R$, or the *radius of the "nine-points" circle of a triangle is one-half the radius of the circumcircle.*

90. *To find the angular distances of the pole of Dr. Hart's Circle from the vertices of the given triangle.*

From the triangle NO_2O_3 (fig. 34), by aid of Art. 40, we have

$$\cos(\rho + r_2)\sin AO_3 + \cos(\rho + r_3)\sin AO_2 = \cos AN \sin O_2O_3. \qquad (1)$$

Also by Art. 55 (1), it follows that

$$\sin r_2 \sin AO_3 + \sin r_3 \sin AO_2 = \frac{2n}{\sin a}\sin O_2O_3, \qquad (2)$$

[See Art. 36, Ex. 2 (a).]

and

$$\sin AO_2 : \sin AO_3 = \sin r_2 : \sin r_3. \qquad (3)$$

Dividing (1) by (2), and substituting the ratios given in (3); therefore

$$\frac{\cos(\rho + r_2)\sin r_3 + \cos(\rho + r_3)\sin r_2}{2\sin r_2 \sin r_3} = \frac{\sin a \cos AN}{2n}.$$

Performing the division on the left-hand side of this equation, we have

$$\cos AN = \frac{n\cos\rho}{\sin a}\left[\cot r_2 + \cot r_3 - 2\tan\rho\right]$$

$$= \frac{\cos^2\rho}{\sin a}\left[\sin(s-b) + \sin(s-c) - 2\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c.\right]$$

[Arts. 79 and 83 (4).]

Hence

$$\cos AN = \frac{\cos\frac{1}{2}b\cos\frac{1}{2}c}{\cos\frac{1}{2}a}\cos\rho,$$

with similar values for $\cos BN$ and $\cos CN$.

Example.

$$\cos AN : \cos BN : \cos CN = \sec^2\tfrac{1}{2}a : \sec^2\tfrac{1}{2}b : \sec^2\tfrac{1}{2}c.$$

91. *The tangents t_1, t_2, t_3, from the vertices of a triangle to Dr. Hart's Circle are given by the equations*

$$\cos t_1 \cos \frac{a}{2} = \cos \frac{b}{2} \cos \frac{c}{2};$$

$$\cos t_2 \cos \frac{b}{2} = \cos \frac{c}{2} \cos \frac{a}{2};$$

$$\cos t_3 \cos \frac{c}{2} = \cos \frac{a}{2} \cos \frac{b}{2};$$

where a, b, c are the sides of the triangle.

For $$\cos AN = \cos \rho \cos t_1.$$

Hence $$\cos t_1 = \frac{\cos AN}{\cos \rho} = \frac{\cos \frac{1}{2} b \cos \frac{1}{2} c}{\cos \frac{1}{2} a}. \qquad \text{(Art. 90.)}$$

92. *To find the angular distances of the points of intersection of the circle with the sides from the vertices of the triangle.*

Let λ and μ be the distances on the side c from the vertex A; then the equation

$$\cos AN \sin (c - \lambda) + \cos BN \sin \lambda = \cos \rho \sin c, \qquad (1)$$

expanded as a quadratic for $\tan \frac{1}{2} \lambda$, will have for its roots the values of $\tan \frac{1}{2} \lambda$ and $\tan \frac{1}{2} \mu$. (Cf. Art. 43.)

Suppose $t = \tan \frac{\lambda}{2}$, the equation (1) readily transforms into

$$\sin c (\cos AN + \cos \rho)\, t^2 + 2 (\cos AN \cos c - \cos BN)\, t$$
$$- (\cos AN - \cos \rho) \sin c = 0;$$

the roots of which, on reduction, are

$$t = \frac{-\cos AN \cos c + \cos BN \pm (\cos AN - \cos BN)}{\sin c (\cos AN + \cos \rho)}$$

$$= \frac{-\cos AN (1 + \cos c) + 2 \cos BN}{\sin c (\cos AN + \cos \rho)}, \text{ or } \frac{\cos AN (1 - \cos c)}{\sin c (\cos AN + \cos \rho)};$$

or by putting in the values of cos AN and cos BN given in Art. 90, we have

$$\tan \tfrac{1}{2}\lambda = \frac{\cos \frac{1}{2} a - \cos \frac{1}{2} b \cos \frac{1}{2} c}{\cos \frac{1}{2} b \sin \frac{1}{2} c}, \tag{2}$$

$$\tan \tfrac{1}{2}\mu = \frac{\cos \frac{1}{2} b \sin \frac{1}{2} c}{\cos \frac{1}{2} a + \cos \frac{1}{2} b \cos \frac{1}{2} c}.$$

93. *To find the intercepts made by Dr. Hart's Circle on the sides of the given triangle.*

From Art. 92 (2), we have directly the value of tan $\frac{1}{2}(\lambda - \mu)$ in terms of a, b, c.

Otherwise thus :—

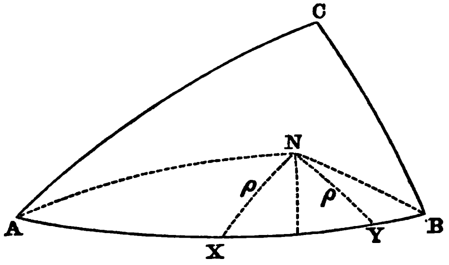

Fig. 35.

Let ABC be the given triangle (fig. 35), N the pole of Dr. Hart's Circle, α a perpendicular on AB from N, $NX = NY = \rho$, the angular radius of the circle, and XY the intercept made by the circle on the side c.

Then, by Art. 90,

$$\cos AN = \frac{\cos \frac{1}{2} b \cos \frac{1}{2} c}{\cos \frac{1}{2} a} \cos \rho = \frac{K}{\cos^2 \frac{1}{2} a},$$

where $K = \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c \cos \rho$, with similar values for cos BN and cos CN.

Also, by p. 38, Ex. 3,

$$\sin^2 c \cos^2 \alpha = \cos^2 AN + \cos^2 BN - 2 \cos AN \cos BN \cos c;$$

or, by substituting the values of cos AN and cos BN given above,

$$\sin^2 c \cos^2 \alpha = K^2 \left[\frac{1}{\cos^4 \frac{1}{2} a} + \frac{1}{\cos^4 \frac{1}{2} b} - \frac{2\,(1 - 2\sin^2 \frac{1}{2} c)}{\cos^2 \frac{1}{2} a \cos^2 \frac{1}{2} b}\right],$$

which gives, by reduction,

$$\sin^2 c \cos^2 \alpha = \frac{\cos^2 \frac{1}{2} a \cos^2 \frac{1}{2} b \cos^2 \frac{1}{2} c \cos^2 \rho\, (\cos^2 \frac{1}{2} a - \cos^2 \frac{1}{2} b)^2}{\cos^4 \frac{1}{2} a \cos^4 \frac{1}{2} b} + \sin^2 c \cos^2 \rho;$$

dividing this equation by $\sin^2 c \cos^2 \rho$, and transposing the last term,

$$\frac{\cos^2 \alpha}{\cos^2 \rho} - 1 = \frac{(\cos^2 \frac{1}{2} a - \cos^2 \frac{1}{2} b)^2}{4 \cos^2 \frac{1}{2} a \cos^2 \frac{1}{2} b \sin^2 \frac{1}{2} c};$$

but

$$\frac{\cos^2 \alpha}{\cos^2 \rho} - 1 = \sec^2 \tfrac{1}{2} XY - 1 = \tan^2 \tfrac{1}{2} XY.$$

Hence

$$\tan \tfrac{1}{2} XY = \frac{\cos^2 \frac{1}{2} a - \cos^2 \frac{1}{2} b}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \sin \frac{1}{2} c}.$$

94. **Analogous Theorem in Plano.**

Since
$$\tan \tfrac{1}{2} XY = \frac{\cos b \sim \cos a}{4 \cos \frac{1}{2} a \cos \frac{1}{2} b \sin \frac{1}{2} c};$$

in a plane triangle the intercepts made by the "nine-points" circle on the sides a, b, c of a triangle are, respectively,

$$\frac{b^2 - c^2}{2a}, \quad \frac{c^2 - a^2}{2b}, \quad \frac{a^2 - b^2}{2c}.$$

95. *To find the angles of intersection of Dr. Hart's Circle with the sides of the given triangle.*

From fig. 35, the angle of intersection of the circle with the side c is obviously the complement of NXY or of NYX.

Let
$$\theta = NXY;$$

then
$$\tan \tfrac{1}{2} XY = \tan \rho \cos \theta.$$

Substituting the values of $\tan \frac{1}{2} XY$ (Art. 93) and $\tan \rho$ (Art. 88) in this equation, we have

$$\frac{\cos b \sim \cos a}{4 \cos \frac{1}{2} a \cos \frac{1}{2} b \sin \frac{1}{2} c} \cdot \frac{n}{\sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c} = \cos \theta.$$

Hence $$\frac{2n\,(\cos b \sim \cos a)}{\sin a \sin b\,(1-\cos c)} = \frac{\cos b \sim \cos a}{1-\cos c}\,.\,\sin C = \cos\theta,$$

which reduces easily by the aid of Ex. 33, p. 76, to the form

$$\cos\theta = \sin\,(A - B);$$

or—*the angle between Dr. Hart's Circle and a side of the triangle is equal to the difference of the angles of the triangle adjacent to that side;* and the same result is known to hold between the "nine-points" circle and the sides of a plane triangle.

Example.

The sines of the angular distances of the pole of Dr. Hart's Circle from the sides of the given triangle are proportional to

$$\cos\,(B - C),\quad \cos\,(C - A),\quad \cos\,(A - B).$$

NOTES.—Expressions have been obtained in Art. 92 for the values of the angular distances from the vertices of the given triangle of the points of intersection of Dr. Hart's Circle with the sides. These formulae may also be proved from the equations

$$\tan\tfrac{1}{2}\lambda\,\tan\tfrac{1}{2}\mu = \frac{\cos\frac{1}{2}a - \cos\frac{1}{2}b\cos\frac{1}{2}c}{\cos\frac{1}{2}a + \cos\frac{1}{2}b\cos\frac{1}{2}c}, \qquad (1)$$

and

$$\tan\frac{c-\lambda}{2}\,\tan\frac{c-\mu}{2} = \frac{\cos\frac{1}{2}b - \cos\frac{1}{2}c\cos\frac{1}{2}a}{\cos\frac{1}{2}b + \cos\frac{1}{2}c\cos\frac{1}{2}a}, \qquad (2)$$

these equations being obtained by the use of Arts. 43 and 91.

Or thus: by Art. 93—

$$\tan\frac{\lambda-\mu}{2} = \frac{\cos a - \cos b}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\sin\frac{1}{2}c}; \qquad (3)$$

and, as before,

$$\tan\tfrac{1}{2}\lambda\,\tan\tfrac{1}{2}\mu = \frac{\cos\frac{1}{2}a - \cos\frac{1}{2}b\cos\frac{1}{2}c}{\cos\frac{1}{2}a + \cos\frac{1}{2}b\cos\frac{1}{2}c}; \qquad (4)$$

and the solutions of either set of simultaneous equations (1) and (2), and (3) and (4), for $\tan\frac{1}{2}\lambda$ and $\tan\frac{1}{2}\mu$, are here left as Exercises for the Student.

Also, by using the theorem of Art. 43, and writing down the values of λ and μ already obtained, and the corresponding values of λ' and μ' made by the circle on the side b measured from A, we find that

$$\tan \tfrac{1}{2}\lambda \tan \tfrac{1}{2} c = \tan \tfrac{1}{2}\lambda' \tan \tfrac{1}{2} b, \qquad (5)$$

and

$$\tan \tfrac{1}{2}\mu' \tan \tfrac{1}{2} c = \tan \tfrac{1}{2}\mu \tan \tfrac{1}{2} b. \qquad (6)$$

From (5) it follows, as *in plano*, that "*the extremities of any side and a pair of corresponding intersections of Dr. Hart's Circle on the other two sides are concyclic.*"

Again, from (6) we have *in plano* $\mu' : \mu = b : c$, or "*a chord of the 'nine-points' circle is parallel to the base.*"

Miscellaneous Examples.

1. Prove that

$$\cot r + \cot r_1 + \cot r_2 + \cot r_3 = 2 \tan R,$$
$$\cot r - \cot r_1 + \cot r_2 + \cot r_3 = 2 \tan R_1,$$

&c., &c.

(Cf. Art. 80, Ex. 5.)

2. The angular radius of the incircle being 45°; prove that the triangle formed by joining its points of contact with the sides is in all respects, save position, the polar triangle.

[Apply Art. 82, (1) and (2).]

3. Prove that

$$(\alpha) \quad \tan R + \tan R_1 + \tan R_2 + \tan R_3 = 2 \cot r$$
$$(\beta) \quad \tan R - \tan R_1 + \tan R_2 + \tan R_3 = 2 \cot r_1,$$

&c., &c.

4. Prove that

$$\tan \tfrac{1}{2} A \sin (s - a) = \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B \tan \tfrac{1}{2} C \sin s$$
$$= \frac{N}{2 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C}.$$

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5. In any triangle,

$$\frac{\tan r}{\tan R} = \frac{\cos (S - A) \cos (S - B) \cos (S - C)}{2 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C},$$

and write down the analogous result *in plano*.

Ans. $\dfrac{r}{R} = 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C.$

*6. Having given the base (c), and the sum of the base angles of a triangle; prove that the length of the arc of the great circle perpendicular to the base at its middle point, and intercepted by the bisectors of the vertical angle, is of constant value.

[Let the arc meet the internal and external bisectors of the vertical angle in M and N, and the base in P; then, by Art. 55 (2), we have

$$2\cos\tfrac{1}{2}(a-b)\tan MP = \tan r_3 - \tan r, \qquad (1)$$

and

$$2\cos\tfrac{1}{2}(a+b)\tan NP = \tan r_1 + \tan r_2. \qquad (2)$$

Hence, since $\tan MN = \tan(MP + NP)$, by substituting the values given in (1) and (2), we get, with the aid of Gauss' formulae,

$$\tan MN = \frac{2\sin\frac{1}{2}c}{\sin(A+B)\cos^2\frac{1}{2}C}].$$

7. Prove the following:—

$$\cot(s-b)\cot(s-c) + \cot(s-c)\cot(s-a) + \cot(s-a)\cot(s-b) = \operatorname{cosec}^2 r.$$

[For $\operatorname{cosec}^2 r = 1 + \dfrac{\sin^2 s}{n^2} = 1 + \dfrac{\sin s}{\sin(s-a)\sin(s-b)\sin(s-c)}$

$$= 1 + \frac{\sin \overline{s-a} + \overline{s-b} + \overline{s-c}}{\sin(s-a)\sin(s-b)\sin(s-c)}\text{; etc., etc.]}$$

8. $\cot(s-b)\cot(s-c) - \cot s\cot(s-b) - \cot s\cot(s-c) = \operatorname{cosec}^2 r_1,$

$\cot(s-c)\cot(s-a) - \cot s\cot(s-c) - \cot s\cot(s-a) = \operatorname{cosec}^2 r_2,$

etc., etc.

[Similar to Ex. 7.]

9. $$\frac{\cot(s-a)}{\sin^2 r_1} + \frac{\cot(s-b)}{\sin^2 r_2} + \frac{\cot(s-c)}{\sin^2 r_3} + \frac{2\cot s}{\sin^2 r}$$

$$= 3\cot(s-a)\cot(s-b)\cot(s-c).$$

---

* Analogous theorem *in plano*:— $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} =$ diameter of the circumcircle.

10.
$$\operatorname{cosec}^2 r_1 + \operatorname{cosec}^2 r_2 + \operatorname{cosec}^2 r_3 - \operatorname{cosec}^2 r = -2 \cot s \{\cot (s - a) + \cot (s - b) + \cot (s - c)\}.$$

[Apply Exs. 7 and 8.]

11.
$$\frac{1}{\sin^2 r} + \frac{1}{\sin^2 r_1} + \frac{1}{\sin^2 r_2} + \frac{1}{\sin^2 r_3} = \frac{-2\Sigma \tan (s-a) \tan (s-b)}{\tan s \tan (s-a) \tan (s-b) \tan (s-c)}.$$

[From Ex. 7,

$$\tan (s-a) \tan (s-b) \tan (s-c) \operatorname{cosec}^2 r = \tan (s-a) + \tan (s-b) + \tan (s-c);$$

and by Ex. 8,

$$\tan s \tan (s-b) \tan (s-c) \operatorname{cosec}^2 r_1 = \tan s - \tan (s-b) - \tan (s-c),$$

with similar values for $\operatorname{cosec}^2 r_2$ and $\operatorname{cosec}^2 r_3$; therefore, &c.]

12.
$$\cot R - \cot R_1 - \cot R_2 - \cot R_3 = \frac{2N \cos s}{n}.$$

[For $\cot R - \cot R_1 - \ldots - \ldots = \cot R \{1 - \tan R (\cot R_1 + \ldots + \ldots)\}$ which, by Art. 84, Ex. 7,

$$= \cot R (1 - \tan \tfrac{1}{2} b \tan \tfrac{1}{2} c - \ldots - \ldots) = \frac{\cot R \cos s}{\cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c} = \text{etc., etc.}]$$

13. If δ be the angular distance between the poles of the incircle and circumcircle of a triangle, r and R their angular radii; prove that

$$\sec^2 R \sec^2 r \sin^2 \delta = \tan^2 R - 2 \tan R \tan r.$$

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[Apply Art, 87, Ex. (1).]

14. In a plane triangle, the distances of the centre of the incircle from the centres of the excircles are—

$$\frac{a}{\cos \frac{1}{2} A}, \quad \frac{b}{\cos \frac{1}{2} B}, \quad \frac{c}{\cos \frac{1}{2} C};$$

and the distances of the centres of the excircles from each other are—

$$\frac{a}{\sin \frac{1}{2} A}, \quad \frac{b}{\sin \frac{1}{2} B}, \quad \frac{c}{\sin \frac{1}{2} C};$$

state and prove the analogous results on the sphere.

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CHAPTER VII.

AREAS.

SECTION I.

Area and Volume of a Sphere.

96. It has been assumed in Art. 8 that the area of a sphere of radius r is $4\pi r^2$. As the areas of spherical figures are of prime importance in the practical applications of Spherical Trigonometry, we shall devote the present Chapter to their determination, and to the development of formulae expressing the areas, or functions of the areas, of figures as functions of their parts. The demonstrations given are those generally adopted in text-books on the subject, but in many cases simple and direct geometrical deductions are also appended.

Definitions:—(1) A plane cutting a sphere divides it into two parts, having circular bases (Art. 9). These parts we propose to call *circular segments* of the sphere; and in speaking of the segment which any plane cuts off we shall, unless otherwise stated, refer to the smaller segment. The more general term, *segment of a sphere*, we reserve for a portion of a sphere cut off by any surface other than a plane.

(2) The cone joining the centre of a sphere to any figure on its surface contains a portion of the sphere which

we call a *sector*. If the figure be a small circle, the cone will be a right circular cone, and we term the sector a *circular sector*.

Reference to Fig. 1 will show how a circular segment and a circular sector may be generated. The segment cut off by the small circle CC' is generated by the revolution of the segment CAC' of the great circle CBC' round the axis AD; and the sector cut out by the cone joining O to the circle CC' is generated by the revolution of the sector OCA of the great circle ACB round the axis OA.

97. **Area of a Circular Segment of a Sphere.**—Let PAB be a great circle of radius r passing through P, the pole of the segment cut off by the small circle AB.

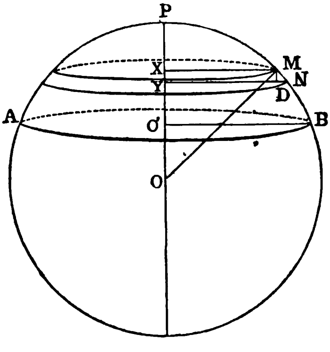

Fig. 36.

Let M and N be two points very near to each other on the circle PAB; MX and NY perpendiculars on OP; and MD a perpendicular on NY. Now, if the arc PB be made to revolve round PO, it will trace out the surface of the segment, and the arc MN will trace out a circular band of it. The width of this band being MN, and its length $2\pi\ MX$ ($MX = NY$ approx.), its area is $2\pi\ MX \,.\, MN$. But MN being very small, may be regarded as straight; and from the similar triangles MND and MOX we have

$$MN : MD = OM : MX;$$

therefore

$$MN \,.\, MX = OM \,.\, MD = r \,.\, XY.$$

Hence the area of the band described by MN is

$$2\pi r \,.\, XY;$$

and the sum of the areas of all the bands in the surface of the segment—that is, the area of the segment—is

$$2\pi r \,.\, PO';$$

but $$PO' = r - OO' = r\,(1 - \cos a),$$

where a is the angle POA, or the spherical radius of the segment.

Therefore the area of the segment is

$$2\pi r^2\,(1 - \cos a) = 4\pi r^2 \sin^2 \tfrac{1}{2}\, a = \pi\,(\text{chord } AP)^{2*}.$$

Examples.

1. The area of a hemisphere is $2\pi r^2$, and the area of a sphere is $4\pi r^2$.

2. The area of a segment of spherical radius a is to the area of the sphere from which it is cut as versin a is to 2.

3. The area of a sphere is equal to that of the circumscribing cylinder.

---

* Hence the area of the segment is equal to the area of a circle of radius AP.

98. **Volume of a Sphere.**—Consider a small cone having its vertex at the centre of the sphere, and its base a small element of the surface. The volume of this cone is found by multiplying the area of its base by one-third of its height; the height being equal to the radius of the sphere. Now we may conceive the sphere to be made up of an infinite number of such cones, and, by adding them together, we find—

$$\begin{aligned}\text{Volume of sphere} &= \tfrac{1}{3} r\ (\text{area of sphere}) \\ &= \tfrac{1}{3} r\ 4\pi r^2. \\ &= \tfrac{4}{3}\pi r^3.\end{aligned}$$

Cor. 1.—The volume of a spherical sector

$$= \tfrac{1}{3} r\ (\text{area of sector}).$$

Cor. 2.—The volume of a circular sector of spherical radius a

$$= \tfrac{2}{3}\pi r^3 (1 - \cos a).$$

99. **Volume of a Circular Segment of a Sphere.**—Join the contour of the base of the segment to the centre of the sphere. The joining lines will form a cone, which with segment will constitute a circular sector. Hence we have—

$$\text{Volume of sector} = \tfrac{2}{3}\pi r^3 (1 - \cos a).$$

$$\text{Volume of cone} = \tfrac{1}{3}\,(\text{area of base})\ (\text{height})$$

$$= \tfrac{1}{3}\,(\pi r^2 \sin^2 a)\ (r \cos a) = \tfrac{1}{3}\pi r^3 \sin^2 a \cos a;$$

therefore volume of segment = sector − cone

$$= \frac{\pi r^3}{3}\,(4 \sin^2 \tfrac{1}{2} a - \sin^2 a \cos a) = \tfrac{4}{3}\pi r^3 \sin^2 \tfrac{1}{2} a\,(1 - \cos^2 \tfrac{1}{2} a \cos a)$$

$$= \tfrac{4}{3}\pi r^3 \sin^4 \tfrac{1}{2} a\ (1 + 2 \cos^2 \tfrac{1}{2} a).$$

SECTION II.

Areas of Triangles.

100. **Problem.**—*To find the area of a spherical triangle, having given the three angles.*

Let ABC (fig. 37) be a spherical triangle, and let the sides BA and CA meet the side BC again in B' and C'.

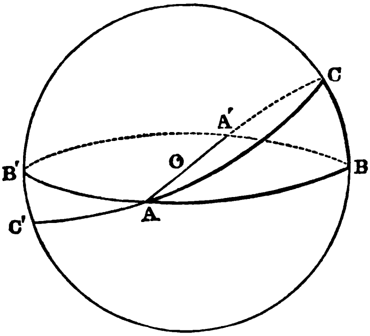

Fig. 37.

It is evident that the triangle $AB'C'$ is equal in every respect to the triangle $A'BC$, which is the colunar of ABC on the side BC. For $A'C = \pi - AC = AC'$, and $A'B = \pi - AB = AB'$, and the angles at A and A' are equal.

Hence $ABC + AB'C' = \text{lune } ACA'B = 2Ar^2$; (Art. 8.)

also $ABC + AB'C = \text{lune } ABCB' = 2Br^2$;

and $ABC + ABC' = \text{lune } CBC'A = 2Cr^2$.

By addition, we get

$$2\,ABC + \text{area of hemisphere} = 2r^2\,(A + B + C),$$

or
$$ABC + \pi r^2 = r^2\,(A + B + C);$$

therefore

$$\text{Area of } ABC = (A + B + C - \pi)\,r^2 = Er^2,$$

where E is the spherical excess; that is, the excess of the sum of the angles of ABC over the sum of the angles of a plane triangle. (*See* concluding paragraph of Art. 29).

Cor.—The areas of the colunar triangles are—

$$(\pi + A - B - C)\,r^2,\ (\pi + B - C - A)\,r^2,\ (\pi + C - A - B)\,r^2,$$

or
$$(2A - E)\,r^2,\ (2B - E)\,r^2,\ (2C - E)\,r^2.$$

Note.—The equality of the areas of the triangles $AB'C'$ and $A'BC$ is evident, from the consideration that one is diametrically opposite to the other; and hence, to every element of one will correspond an equal element of the other. Also the triangles subtend equal solid angles at the centre of the sphere. These triangles, although not capable of direct superposition, can be divided into triangles which are capable of superposition, by joining the poles of their circumcircles to the vertices of the triangles.

101. **Problem.**—*Given the base and area of a spherical triangle, to find the locus of its vertex* (Lexell).

In this case, since we are given the area we know the sum of the angles, or S. Hence, referring to Art. 20, Ex. 5, it follows that the locus of the vertex is a circle, of which O (fig. 10) is the pole. The point O is the pole of the circumcircle of the colunar triangle on the base AB. *In plano* it is removed to infinity, and the locus becomes a right line parallel to the given base.

Again, since O is the pole of the circumcircle of the colunar on AB; and since OAB $(= \pi - S)$ is given, O is fixed, and the vertex C' of the colunar describes a small circle passing through A and B; but since C is diametrically opposite to C', it follows that C describes a small circle passing through the points diametrically opposite to A and B. (Cf. Art. 82, Ex. 3.)

Examples.

1. If E_1, E_2, E_3 denote the spherical excesses of the colunars on a, b, c, respectively, show that

$$E + E_1 + E_2 + E_3 = 2\pi;$$

and therefore the sum of the areas of any triangle and its colunars is half the area of the sphere.

*2. Show that the area of a spherical triangle is to that of half the surface of the sphere on which it is described as the spherical excess is to four right angles.

(a) A, B, C are three angular points of a cube inscribed in a sphere, no two of the points being on the same edge of the cube; show that the area of the spherical triangle ABC is one-fourth of the area of the sphere.

[The chordal triangle is equilateral, and its side

$$= \sqrt{\frac{8r^2}{3}}. \quad \text{Hence } \sin \tfrac{1}{2} a = \sqrt{\frac{2}{3}}, \text{ and } A = \frac{2\pi}{3}; \text{ therefore, \&c.}$$

(Art. 27, Ex. 8.) The angle between two sides, being the angle between the planes joining the centre to the diagonals of two faces meeting at corner, is easily seen by symmetry to be 120°.]

3. Find the area of an equilateral triangle, each side being 60° on a sphere of 6 inches radius.

[By Art. 27, Ex. 8, $\sec A = 3$;

therefore $A = 70^\circ\ 31'\ 43\tfrac{3}{5}''$. $E = 31^\circ\ 35'\ 10\tfrac{4}{5}'' = 113711''$.

Hence $$\text{Area} = \frac{113711}{206265}\, 36 = 19{\cdot}845 \text{ sq. inches.}]$$

4. The area of a triangle, each of the sides of which is a quadrant, is one-eighth of the surface of the sphere.

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* *Science and Art Exam. Papers.*

*5. If the sum of one pair of opposite angles of a variable spherical quadrilateral inscribed in a small circle is constant, the area remains constant. [Cf. Art. 19 (4).]

102. In Chapter IV. we have seen that when any three parts of a spherical triangle are given, the triangle can be completely solved. We may therefore find the area from the formula, Area $= Er^2$, when the angles of the triangle not included in the data have been first determined.

We proceed to show in the following Articles how to calculate the area, having given—

(1) The three sides;

(2) Two sides and the included angle,

without resorting to the direct calculation of the value of each of the angles.

103. **Problem.**—*To find the area of a spherical triangle, having given the three sides.* The object here is to express E in terms of the sides.

I. **Cagnoli's Method :—**

$$\sin \tfrac{1}{2} E = \sin \tfrac{1}{2}(A+B+C-\pi)$$

$$= \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(C-\pi) + \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(C-\pi)$$

$$= \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2} C - \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2} C$$

$$= \frac{\sin \frac{1}{2} C \cos \frac{1}{2} C}{\cos \frac{1}{2} c}\left[\cos \tfrac{1}{2}(a-b) - \cos \tfrac{1}{2}(a+b)\right] \quad \text{(Art. 64)}$$

$$= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b \sin C}{\cos \frac{1}{2} c} = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \cdot \frac{2n}{\sin a \sin b}$$

$$= \frac{n}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}.$$

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Examples.

1. Given $a = 46° \; 24'$, $b = 67° \; 14'$, $c = 81° \; 12'$; find the area of the triangle, the radius of the sphere being r.

$$\log \sin \tfrac{1}{2} E = \log \frac{n}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}.$$

$$L \sin \tfrac{1}{2} E = 9{\cdot}4518482;$$

therefore $$E = 32° \; 53' = 118380'';$$

therefore $$\text{Area} = \frac{118380}{206265} r^2.$$

2. Given $a = 108° \; 14'$, $b = 75° \; 29'$, $c = 56° \; 37'$.
Ans. $E = 48° \; 32' \; 34{\cdot}5''$.

3. Given $a = \; 57° \; 17'$, $b = 20° \; 39'$, $c = 76° \; 22'$.
Ans. $E = 5° \; 11' \; 24{\cdot}5''$.

4. Given $a = \; 63° \; 54'$, $b = 47° \; 18'$, $c - 53° \; 26'$.
Ans. $E = 24° \; 29' \; 49\tfrac{1}{2}''$.

II. **Lhuilier's Method.**—Lhuilier obtained the following expression for $\tan \frac{1}{4} E$:—

$$\tan \tfrac{1}{4} E = \frac{\sin \frac{1}{4} (A + B + C - \pi)}{\cos \frac{1}{4} (A + B + C - \pi)}$$

$$= \frac{\sin \frac{1}{2} (A + B) - \sin \frac{1}{2} (\pi - C)}{\cos \frac{1}{2} (A + B) + \cos \frac{1}{2} (\pi - C)}$$

$$= \frac{\sin \frac{1}{2} (A + B) - \cos \frac{1}{2} C}{\cos \frac{1}{2} (A + B) + \sin \frac{1}{2} C}$$

$$= \frac{\cos \frac{1}{2} (a - b) - \cos \frac{1}{2} c}{\cos \frac{1}{2} (a + b) + \cos \frac{1}{2} c} \cdot \frac{\cos \frac{1}{2} C}{\sin \frac{1}{2} C} \qquad \text{(Art. 64.)}$$

$$= \frac{\sin \frac{1}{2} (s - b) \sin \frac{1}{2} (s - a)}{\cos \frac{1}{2} s \cos \frac{1}{2} (s - c)} \sqrt{\frac{\sin s \, . \, \sin (s - c)}{\sin (s - a) \sin (s - b)}}$$

$$= \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2} (s - a) \tan \tfrac{1}{2} (s - b) \tan \tfrac{1}{2} (s - c)}.$$

104. **Geometrical Methods.**—Referring to Art. 20, we see that if X, Y, Z be the middle points of the sides of a triangle, AL a perpendicular from A on XY; then (fig. 38) $AD = \frac{1}{2}(\pi - c)$, $LD = \frac{1}{2}\pi - \gamma$ where $\gamma = XY$, and $LAD = \pi - S = \frac{1}{2}(\pi - E)$.

Therefore, since

$$\cos \tfrac{1}{2} AD = \cot D \cot LAD,$$

we have

$$\sin \tfrac{1}{2} c = \cot D \tan \tfrac{1}{2} E. \qquad (1)$$

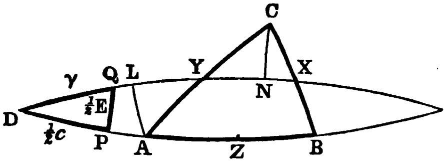

Fig. 38.

This equation shows that if DP be taken equal to $\frac{1}{2}c$; and a perpendicular PQ be drawn to AD at P; then

$$PQ = \tfrac{1}{2} E. \qquad (2)$$

Again, since

$$\sin DL = \sin AD \sin LAD,$$

we have

$$\cos \gamma = \cos \tfrac{1}{2} c \cos \tfrac{1}{2} E;$$

and, therefore,

$$DQ = \gamma. \qquad (3)$$

Thus the sides of the triangle PQD are $\frac{1}{2}c$, $\frac{1}{2}E$, and γ, respectively.

Therefore, from it we find

$$\sin \tfrac{1}{2} E = \sin \gamma \sin D = \sin \gamma \frac{\sin AL}{\cos \frac{1}{2} c} = \frac{\sin \gamma \sin CN}{\cos \frac{1}{2} c}$$

$$= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b \sin C}{\cos \frac{1}{2} c} = \frac{n}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}.$$

Cor.—Since

$$\cos DQ = \cos PQ \,.\, \cos PD,$$

we have

$$\cos \tfrac{1}{2} E \cos \tfrac{1}{2} c = \cos \gamma = \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \cos C.$$

105. **Problem.**—*To find the area of a spherical triangle, having given two sides and the included angle.*

From the triangle PQD (fig. 38), we have

$$\cos \gamma = \cos \tfrac{1}{2} c \cos \tfrac{1}{2} E,$$

and, by the foregoing,

$$\sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin C = \cos \tfrac{1}{2} c \sin \tfrac{1}{2} E;$$

therefore, by division, we find

$$\tan \tfrac{1}{2} E = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b \sin C}{\cos \frac{1}{2} a \cos \frac{1}{2} b + \sin \frac{1}{2} a \sin \frac{1}{2} b \cos C}, \qquad (1)$$

the denominator in this expression being equal to $\cos \gamma$.

Again, since

$$LAD = \tfrac{1}{2} (\pi - E),$$

we have

$$LAY = S - A = \tfrac{1}{2} \pi - (A - \tfrac{1}{2} E),$$

and

$$\tan AL = \tan AD \cos LAD = \tan AY \cos LAY;$$

that is,

$$\cot \tfrac{1}{2} c \sin \tfrac{1}{2} E = \tan \tfrac{1}{2} b \sin (A - \tfrac{1}{2} E);$$

therefore

$$\frac{\sin (A - \frac{1}{2} E)}{\sin \frac{1}{2} E} = \cot \tfrac{1}{2} b \cot \tfrac{1}{2} c. \qquad (2)$$

When two sides and the included angle are given, E may be calculated by means of either of the formulae (1) or (2) of this Article. They are not, however, in a shape suitable for logarithmic computation, and the use of a subsidiary angle becomes necessary. (See Art. 76.)

Remark.—If we denote the spherical excesses of the colunar triangles on the sides a, b, c, by E_1, E_2, E_3, respectively, we have

$$E_1 = 2A - E, \quad E_2 = 2B - E, \quad E_3 = 2C - E,$$

and the equation (2) becomes

$$\frac{\sin \frac{1}{2} E_1}{\sin \frac{1}{2} E} = \cot \tfrac{1}{2} b \cot \tfrac{1}{2} c. \qquad (3)$$

. . . .

Examples.

1. $\dfrac{\sin \frac{1}{2} E_1}{\tan \frac{1}{2} a} = \dfrac{\sin \frac{1}{2} E_2}{\tan \frac{1}{2} b} = \dfrac{\sin \frac{1}{2} E_3}{\tan \frac{1}{2} c} = \dfrac{\sin \frac{1}{2} E}{\tan \frac{1}{2} a \tan \frac{1}{2} b \tan \frac{1}{2} c}.$

2. $\dfrac{\sin \frac{1}{2} E_1}{\tan R_1} = \dfrac{\sin \frac{1}{2} E_2}{\tan R_2} = \dfrac{\sin \frac{1}{2} E_3}{\tan R_3} = \dfrac{\sin \frac{1}{2} E}{\tan R}.$

106. The formula for $\sin \frac{1}{2} E$ deduced in Art. 104 leads to the expression obtained in Art. 97 for the area of a circular segment of a sphere. Consider a small triangle formed by drawing two great circles through the pole P of the segment, very close to each other, meeting the circular base of the segment in a very small arc AB. The area of the triangle PAB is very nearly the same as if AB were an arc of a great circle, and, by taking AB very small, the difference is infinitely diminished. Hence,

$$\sin \tfrac{1}{2} E = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \sin C = \sin^2 \tfrac{1}{2} a \times \sin C;$$

or, since E and C are very small, we have

$$E = 2\sin^2 \tfrac{1}{2} a \,.\, C.$$

Supposing the area of the segment to be made up of an infinite number of such triangles, we have its area equal to their sum; that is,

$$\Sigma E r^2 = 2r^2 \sin^2 \tfrac{1}{2} a \; 2\pi$$
$$= 2\pi r^2 (1 - \cos a).$$

Examples.

1. What is the analogous formula *in plano*—

(1) To Cagnoli's theorem;
(2) To Lhuilier's theorem?

Ans. (1) $\Delta = \frac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}$;
(2) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

2. If C be a right angle; prove that

$$\sin \tfrac{1}{2} E = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c},$$

$$\cos \tfrac{1}{2} E = \frac{\cos \frac{1}{2} a \cos \frac{1}{2} b}{\cos \frac{1}{2} c}.$$

3. Find the area of a spherical triangle in terms of the arcs α, β, γ joining the middle points of its sides. (See fig. 38.)

[From the triangle, PQD, $\sin \frac{1}{2} E = \sin \gamma \sin D$; but $\sin D = \sin ZX \sin ZXY$, since $ZD = 90°$; therefore, if $\alpha + \beta + \gamma = 2\sigma$, we have

$\sin \frac{1}{2} E = \sin \beta \sin \gamma \sin X = 2\sqrt{\sin \sigma \sin(\sigma - \alpha) \sin(\sigma - \beta) \sin(\sigma - \gamma)}$].

4. If $\widehat{a\alpha}$, $\widehat{b\beta}$, $\widehat{c\gamma}$ denote the angles between a and α, b and β, c and γ, respectively; prove that

$$\sin \alpha \sin \widehat{a\alpha} = \sin \beta \sin \widehat{\beta b} = \sin \gamma \sin \widehat{\gamma c}$$
$$= 2n \text{ of triangle } XYZ = \sin \tfrac{1}{2} E.$$

APPROXIMATE FORMULAE.

107. *If the sides of a spherical triangle be very small compared with the radius of the sphere on which it is described, its area is approximately equal to the area of a plane triangle having sides of the same length.*

This may have been already anticipated by the student from the results proved in Art. 29. For we have proved there that

$$A = A' + \frac{\Delta}{3r^2},\quad B = B' + \frac{\Delta}{3r^2},\quad C = C' + \frac{\Delta}{3r^2},$$

where Δ is the area of a plane triangle whose sides are equal in length to those of the spherical triangle; and A', B', C' its angles.

Therefore

$$A + B + C - A' - B' - C' = \frac{\Delta}{r^2};$$

or

$$Er^2 = \Delta,\ \text{since}\ A' + B' + C' = \pi.$$

This result may be obtained directly from Lhuilier's formula for tan $\frac{1}{4}E$, as in the closer approximation of the following Article, or from Cagnoli's or other formula for E.

108. *If r, the radius of a sphere, be very large compared with the lengths α, β, γ of the sides of a triangle described upon it; to show that if Δ be the area of a plane triangle, having sides α, β, γ, the area of the spherical triangle is approximately*

$$\Delta\left(1 + \frac{\alpha^2 + \beta^2 + \gamma^2}{24r^2}\right).$$

We have

$$\tan \tfrac{1}{4} E = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2} (s-a) \tan \tfrac{1}{2} (s-b) \tan \tfrac{1}{2} (s-c)}.$$

But
$$\tfrac{1}{2} s = \frac{\alpha + \beta + \gamma}{4r};$$

therefore, if we write $\alpha + \beta + \gamma = 2\sigma$, we will have $s = \frac{\sigma}{r}$ = a very small quantity.

And

$$\tan \tfrac{1}{2} s = \frac{\tfrac{1}{2} s \left(1 - \frac{s^2}{24} + \ldots\right)}{1 - \frac{s^2}{8} + \ldots} = \frac{\frac{\sigma}{2r}\left(1 - \frac{\sigma^2}{24r^2}\right)}{1 - \frac{\sigma^2}{8r^2}}$$

$$= \frac{\sigma}{2r}\left(1 - \frac{\sigma^2}{24r^2}\right)\left(1 + \frac{\sigma^2}{8r^2}\right) = \frac{\sigma}{2r}\left(1 + \frac{\sigma^2}{12r^2}\right);$$

therefore

$$\tan \tfrac{1}{4} E = \sqrt{\frac{\sigma}{2r} \cdot \frac{\sigma - \alpha}{2r} \cdot \frac{\sigma - \beta}{2r} \cdot \frac{\sigma - \gamma}{2r}\left(1 + \frac{\sigma^2}{12r^2}\right)\left(1 + \frac{\overline{\sigma - \alpha}^2}{12r^2}\right) \ldots}$$

$$\tfrac{1}{4} E = \frac{\Delta}{4r^2}\sqrt{1 + \frac{\sigma^2 + (\sigma - \alpha)^2 + (\sigma - \beta)^2 + (\sigma - \gamma)^2}{12r^2}}.$$

Hence,

$$Er^2 = \Delta\left(1 + \frac{\alpha^2 + \beta^2 + \gamma^2}{24r^2}\right).$$

109. **Roy's Rule.**—*The area of a spherical triangle on the Earth's surface being known, to establish a formula for computing the spherical excess.*

Let A be the area of the triangle; then we have the equation

$$Er^2 = A;$$

but if n be the number of seconds in the spherical excess, we have

$$E = \frac{n}{206265};$$

therefore

$$A = \frac{nr^2}{206265}.$$

Now, the length of a degree on the Earth's surface is found to be 365155 feet: thus,

$$\frac{\pi r}{180} = 365155.$$

Substituting this value of r in the former equation, we find

$$\log n = \log A - 9{\cdot}3267737,$$

where A is expressed in square feet.

This formula is called General Roy's rule, as it was used by him in the Trigonometrical Survey of the British Isles. He gave it in the following form:—"*From the logarithm of the area of the triangle, taken as a plane one, in feet, subtract the constant logarithm* 9·3267737; *and the remainder is the logarithm of the excess above* 180°, *in seconds, nearly.*"

110. **Reduction of an Angle to the Horizon.**—*Given the latitudes and the angular distance between two places, to find the difference of their longitudes if their latitudes be very small.* (Cf. Art. 27, Ex. 10; and Ex. 14, p. 103.)

Let Z be the pole (fig. 39), a and b the places; Aa and

Bb are the latitudes λ and λ' of the places, and *AB* is their difference of longitude. If $ab = \delta$, we have

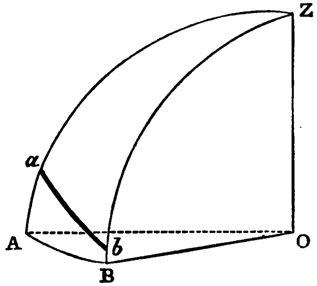

Fig. 39.

$$\cos AB = \cos Z = \frac{\cos\delta - \sin\lambda\sin\lambda'}{\cos\lambda\cos\lambda'}. \qquad (1)$$

This gives the exact value of *AB*. An approximate value may be obtained by assuming it equal to $\delta + x$, where x is essentially small.

Hence, by (1), $$\cos(\delta + x) = \frac{\cos\delta - \lambda\lambda'}{1 - \frac{1}{2}(\lambda^2 + \lambda'^2)}$$

or $$(\cos\delta - x\sin\delta)\left[1 - \tfrac{1}{2}(\lambda^2 + \lambda'^2)\right] = \cos\delta - \lambda\lambda';$$

therefore

$$x\sin\delta = \lambda\lambda' - \tfrac{1}{2}(\lambda^2 + \lambda'^2)\cos\delta, \text{ nearly};$$

and $$x = \frac{2\lambda\lambda' - (\lambda^2 + \lambda'^2)(\cos^2\frac{1}{2}\delta - \sin^2\frac{1}{2}\delta)}{2\sin\delta}$$

$$= \tfrac{1}{4}\left[(\lambda + \lambda')^2\tan\tfrac{1}{2}\delta - (\lambda - \lambda')^2\cot\tfrac{1}{2}\delta\right].$$

If *AOB* be the plane of the horizon, *a* and *b* two objects, the angular distance between which is measured by an observer at *O*, the above correction, x, applied to δ, wlil give the horizontal angle AB. This process may be used in surveying, and is called, *reducing an angle to the horizon.*

SECTION III.

Areas of Polygons—Maximum Areas.

111. **Area of a Polygon.**—*Having given the sum of the angles of a spherical polygon, to find its area.*

Let Σ denote the sum of the angles of the polygon, and n the number of its sides. Take any point within the polygon, and join it to all the angular points; the polygon is thus divided into n triangles, and its area is equal to the sum of the areas of the triangles

$$= (\Sigma + 2\pi - n\pi)\, r^2 = \{\Sigma - (n-2)\,\pi\}\, r^2 = Er^2,$$

where E, the *Spherical Excess* of the polygon, denotes the excess of the sum of the angles of a spherical polygon over that of a plane polygon of the same number of sides.

Cor. 1.—The area of a spherical quadrilateral is

$$(A + B + C + D - 2\pi)\, r^2.$$

Cor. 2.—If a quadrilateral be inscribed in a small circle, $A + C = B + D$; and hence the area of a cyclic quadrilateral is

$$2\,(A + C - \pi)\, r^2.$$

112. **Area of a Quadrilateral.**—The corollaries of the foregoing Article determine the area of a spherical quadrilateral, when the sum of its angles is given; and of a cyclic spherical quadrilateral, when the sum of a pair of opposite angles is given.

If, however, the sides a, b, c, d, and diagonals δ, δ' of any quadrilateral be known, its area may be determined from the equation

$$\sin^2 \tfrac{1}{4} E = \frac{(\sin\tfrac{1}{2}\delta \sin\tfrac{1}{2}\delta' + \cos\tfrac{1}{2}a \cos\tfrac{1}{2}c - \cos\tfrac{1}{2}b \cos\tfrac{1}{2}d)(\sin\tfrac{1}{2}\delta \sin\tfrac{1}{2}\delta' - \cos\tfrac{1}{2}a \cos\tfrac{1}{2}c + \cos\tfrac{1}{2}b \cos\tfrac{1}{2}d)}{4 \cos\tfrac{1}{2}a \cos\tfrac{1}{2}b \cos\tfrac{1}{2}c \cos\tfrac{1}{2}d}$$

where E is the spherical excess of the quadrilateral.

A simple proof of this equation is given in Chap. XIII., Sec. II.

Cor. 1.—When the quadrilateral is cyclic, we have

$$\sin\tfrac{1}{2}\delta \sin\tfrac{1}{2}\delta' = \sin\tfrac{1}{2}a \sin\tfrac{1}{2}c + \sin\tfrac{1}{2}b \sin\tfrac{1}{2}d,$$

and the above formula becomes

$$\sin^2 \tfrac{1}{4} E = \frac{\sin\tfrac{1}{2}(s-a) \sin\tfrac{1}{2}(s-b) \sin\tfrac{1}{2}(s-c) \sin\tfrac{1}{2}(s-d)}{\cos\tfrac{1}{2}a \cos\tfrac{1}{2}b \cos\tfrac{1}{2}c \cos\tfrac{1}{2}d},$$

where $2s = a + b + c + d$.

Cor. 2.—When the quadrilateral is circumscribed to one circle and inscribed in another, we have the additional relation $a + c = b + d$, and hence,

$$\sin^2 \tfrac{1}{4} E = \tan\tfrac{1}{2}a \tan\tfrac{1}{2}b \tan\tfrac{1}{2}c \tan\tfrac{1}{2}d.$$

MAXIMUM AREAS.

113. **Problem.**—*Having given two sides of a spherical triangle, to determine when its area is a maximum.*

Let AB and AC (fig. 38) be the given sides; then the triangle LAD gives

$$\cos LAD = \tan AL \cot AD,$$

or

$$\sin\tfrac{1}{2}E = \tan AL \tan\tfrac{1}{2}c.$$

Now, when $A = 0°$, we have also $AL = 0°$ and $E = 0°$; and, as A increases from zero, AL increases, and $\sin \frac{1}{2} E$ increases; but obviously the greatest value AL can have is $\frac{1}{2} b$; and in this case, since $LAB = S$, we have

$$A = S = B + C,$$

and

$$\sin \tfrac{1}{2} E = \tan \tfrac{1}{2} b \tan \tfrac{1}{2} c.$$

Now E may have any value between 0° and 2π, and $\sin \frac{1}{2} E$ cannot be greater than unity. Hence we have the following cases:—

CASE I.—$b + c < \pi$.

In this case, $\tan \frac{1}{2} (b + c)$ is always positive, and therefore $\tan \frac{1}{2} b \tan \frac{1}{2} c$ is less than unity. Hence, $\sin \frac{1}{2} E$ is always less than unity, and reaches its maximum value, $\tan \frac{1}{2} b \tan \frac{1}{2} c$, when $AL = \frac{1}{2} b$; or when

$$A = B + C.$$

Hence the area of the triangle is a maximum, when the angle between the given sides is equal to the sum of the other two.

Cor. 1.—In this case, the pole of the circumcircle is at the middle point of the side BC; B and C are thus diametrically opposite points on it; *the chordal triangle is therefore right-angled at A, and its area is also a maximum.*[*]

Cor. 2.—The area of the colunar on BC is one-fourth the area of the sphere, or πr^2.

Cor. 3.—As A increases beyond the value which gives the maximum area, the area of ABC diminishes to zero when $A = \pi$, and the colunar on BC is half the area of the sphere.

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[*] See further, Chap. VIII., on the Chordal Triangle.

CASE II.—$b + c = \pi$.

In this case, $\tan \frac{1}{2} b \tan \frac{1}{2} c = 1$, and E increases as AL increases, up to its maximum value, $\frac{1}{2} b$, in which case $\sin \frac{1}{2} E = 1$;

or
$$E = \pi = A = B + C.$$

(See Art. 24.)

Thus the area increases as A increases, up to its extreme limit, ABC, being throughout equal to the colunar on BC, each finally being one-fourth the area of the sphere.

CASE III.—$b + c > \pi$.

In this case, $\tan \frac{1}{2} b \tan \frac{1}{2} c$ is greater than unity; and since $\sin \frac{1}{2} E$ cannot be greater than unity, AL can never be so great as $\frac{1}{2} b$, but will reach the value given by the equation

$$\tan AL \tan \tfrac{1}{2} c = 1, \text{ or } AL = \tfrac{1}{2} (\pi - c).$$

AL will then diminish ($\frac{1}{2} E$ at the same time increasing beyond $\frac{1}{2} \pi$), until it again becomes zero, when $\frac{1}{2} E = \pi$; that is, when the area of the triangle is half the area of the sphere.

It is thus clear that the area of the triangle ABC has no real maximum value when $b + c > \pi$, but increases until the triangle loses its form, and becomes a great circle or hemisphere. However, the colunar triangle on BC satisfies the condition of Case I. and is of maximum area, when

$$A = (\pi - B) + (\pi - C).$$

Cor.—The area of ABC in this case is πr^2, and increases from this value to $2\pi r^2$.

Examples.

1. Deduce the above results from the equation

$$\sin \tfrac{1}{2} E = \sin \tfrac{1}{2} E_1 \tan \tfrac{1}{2} b \tan \tfrac{1}{2} c.$$

2. When two sides of a triangle are given, and its area a maximum, the arcs joining their middle points to the middle point of the base are perpendicular to them, and each meets the opposite side in the pole of the other.

3. If the area of a spherical triangle is one-fourth the area of the sphere, the middle points of its sides are the poles of the circumcircles of its colunars, and the arcs joining the middle points of its sides are quadrants.

114. If a string of given length be laid in any manner on the surface of a sphere, and its extremities joined by an arc of a great circle, the area of the figure thus formed is a maximum when the string is in the form of a semicircle.

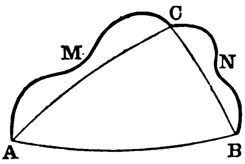

Fig. 40.

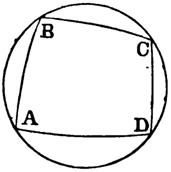

Fig. 41.

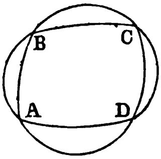

Fig. 42.

Join A and B (fig. 40), the extremities of the string, to any point C on it. By supposing AC and BC to remain constant in length, and the areas AMC and BNC to be rigidly attached to them, we can increase the area of the triangle ABC, and therefore of the whole figure, by making $C = A + B$, and therefore the curve $AMCNB$

must be such that the triangle formed by joining A and B to any point C on it will satisfy the condition $C = A + B$; that is, it must be a semicircle, of which the middle point of AB is the pole.

Cor.—If the ends of the string be united, the area of the figure it forms on a sphere is a maximum when it is a small circle, for each half of it must be a semicircle.

115. **Theorem.**—*If the sides of a polygon be given, its area is a maximum when its vertices lie on a circle.*

For suppose the polygon to be inscribed in a circle (fig. 41), and let it be distorted (fig. 42) so as to take another shape, the portions of the area between the sides and the circumcircle remaining rigidly attached to them; then the new figure will have the same perimeter as the original, and will therefore be of less area (Art. 114, *Cor.*); but the portions on the sides are the same in both; therefore the new polygon is less in area than the original.

Cor.—When the sides of a quadrilateral are given, its area is a maximum when the sum of one pair of opposite angles is equal to the sum of the other pair.

Miscellaneous Examples.

1. Prove that

$$\cos \tfrac{1}{2} E = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c} = \frac{\cos^2 \tfrac{1}{2} a + \cos^2 \tfrac{1}{2} b + \cos^2 \tfrac{1}{2} c - 1}{2 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c}.$$

[$\cos \gamma = \cos \tfrac{1}{2} c \cos \tfrac{1}{2} E$ (fig. 38). Cf. Art. 105; and p. 67, Ex. 4.]

2. Prove that

$$\sin^2 \tfrac{1}{4} E = \frac{\sin \tfrac{1}{2} s \sin \tfrac{1}{2} (s - a) \sin \tfrac{1}{2} (s - b) \sin \tfrac{1}{2} (s - c)}{\cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c}.$$

[Apply Ex. 1.]

3. Prove that

$$\cos^2 \tfrac{1}{4} E = \frac{\cos \frac{1}{2} s \cos \frac{1}{2}(s-a) \cos \frac{1}{2}(s-b) \cos \frac{1}{2}(s-c)}{\cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}.$$

4. From Examples 2 and 3 deduce *Lhuilier's Theorem.*

5. Deduce *Lhuilier's Theorem*, from Ex. 1, and the formula

$$\tan \tfrac{1}{4} E = \frac{1 - \cos \frac{1}{2} E}{\sin \frac{1}{2} E}, \quad \text{or} = \frac{\sin \frac{1}{2} E}{1 + \cos \frac{1}{2} E}.$$

6. Show that

$$\sin(C - \tfrac{1}{2} E) = \frac{n}{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \cos \frac{1}{2} c}.$$

[Apply Art. 103 to colunar; and cf. p. 75, Ex. 31.]

7. Show that

$$\cos(C - \tfrac{1}{2} E) = \frac{1 + \cos c - \cos a - \cos b}{4 \sin \frac{1}{2} a \sin \frac{1}{2} b \cos \frac{1}{2} c} = \cos \tfrac{1}{2} E_3$$

$$= \frac{\cos^2 \frac{1}{2} c - \cos^2 \frac{1}{2} a - \cos^2 \frac{1}{2} b + 1}{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \cos \frac{1}{2} c}.$$

[Cf. Ex. 1. This relation is obviously identical with that of Ex. 1, applied to the colunar triangle on the side c. See also p. 75, Ex. 29.]

8. Prove the relations

$$(\alpha) \quad \sin^2 \tfrac{1}{2}(C - \tfrac{1}{2} E) = \frac{\cos \frac{1}{2} s \sin \frac{1}{2}(s-a) \sin \frac{1}{2}(s-b) \cos \frac{1}{2}(s-c)}{\sin \frac{1}{2} a \sin \frac{1}{2} b \cos \frac{1}{2} c};$$

$$(\beta) \quad \cos^2 \tfrac{1}{2}(C - \tfrac{1}{2} E) = \frac{\sin \frac{1}{2} s \cos \frac{1}{2}(s-a) \cos \frac{1}{2}(s-b) \sin \frac{1}{2}(s-c)}{\sin \frac{1}{2} a \sin \frac{1}{2} b \cos \frac{1}{2} c}.$$

[These relations can be derived from Examples 2 and 3, applied to the colunar on c.]

9. Prove that if E_1, E_2, E_3 be the spherical excesses of the colunar triangles on the sides a, b, c, respectively,

$$\tan^2 \tfrac{1}{4} E_1 = \frac{\tan \frac{1}{2}(s-b) \tan \frac{1}{2}(s-c)}{\tan \frac{1}{2} s \tan \frac{1}{2}(s-a)},$$

with similar expressions for E_2 and E_3.

10. Hence, show that

$$\cot \tfrac{1}{4} E \tan \tfrac{1}{4} E_1 \tan \tfrac{1}{4} E_2 \tan \tfrac{1}{4} E_3 = \cot^2 \tfrac{1}{2} s,$$
$$\tan \tfrac{1}{4} E \cot \tfrac{1}{4} E_1 \tan \tfrac{1}{4} E_2 \tan \tfrac{1}{4} E_3 = \tan^2 \tfrac{1}{2} (s - a),$$
$$\tan \tfrac{1}{4} E \tan \tfrac{1}{4} E_1 \cot \tfrac{1}{4} E_2 \tan \tfrac{1}{4} E_3 = \tan^2 \tfrac{1}{2} (s - b),$$
$$\tan \tfrac{1}{4} E \tan \tfrac{1}{4} E_1 \tan \tfrac{1}{4} E_2 \cot \tfrac{1}{4} E_3 = \tan^2 \tfrac{1}{2} (s - c).$$

(Cf. form with Art. 80, Ex. 2.)

*11. Prove that

$$\sin s = \frac{\sqrt{\sin \tfrac{1}{2} E \sin \tfrac{1}{2} E_1 \sin \tfrac{1}{2} E_2 \sin \tfrac{1}{2} E_3}}{2 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C}.$$

[Supplemental to *Cagnoli's Theorem.* See also *Exam. Paper* XII., Ex. 2.]

12. If the sum of the angles of a triangle be four right angles; prove that

$$\cos^2 \tfrac{1}{2} a + \cos^2 \tfrac{1}{2} b + \cos^2 \tfrac{1}{2} c = 1.$$

[Apply Ex. 1.]

13. If two sides of a triangle, a and b, be supplementary, prove that the spherical excess is equal to the included angle C; and if E' denote the spherical excess of the polar triangle, prove that

$$\sin \tfrac{1}{2} E' = \sin a \cos \tfrac{1}{2} C.$$

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[$A + B = \pi$, by Art. 24; therefore $E = C$. Also, $E' = \pi - c$, and $\cos c = - \cos^2 a + \sin^2 a \cos C$, and $\cos \tfrac{1}{2} c = \sin \tfrac{1}{2} E'$.]

14. Show that

$$\sin^2 \tfrac{1}{2} E = \frac{\sqrt{\sin \tfrac{1}{2} E \sin \tfrac{1}{2} E_1 \sin \tfrac{1}{2} E_2 \sin \tfrac{1}{2} E_3}}{\cot \tfrac{1}{2} a \cot \tfrac{1}{2} b \cot \tfrac{1}{2} c}.$$

[Apply Art. 105 (2).]

15. Two spherical triangles have equal vertical angles, and are such that the products of the tangents of the halves of the sides about the equal angles are equal to one another; prove that the areas of the triangles are the same.

(See Art. 105.)

What does this theorem degenerate to on a plane?

(*Euc.*, VI. xv.)

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* Hence the area of the polar triangle in terms of the angles of the primitive triangle.

16. Through one of the vertices of a triangle draw an arc of a great circle, bisecting its area.

[Let the arc be drawn through the vertex C, meeting the base in a point C'. Apply Art. 105 (2) to the triangles ACC' and BCC'. The arc CC' divides the base into segments, the sines of the halves of which are in the ratio $\cos \frac{1}{2} a : \cos \frac{1}{2} b$].

17. Through a given point on a side of a spherical triangle draw an arc of a great circle, cutting off a given part of the triangle.

[Art. 105 (2) gives the point where the arc meets another side; therefore, &c.].

18. If α, β, γ be the radii of three small circles, which touch each other at $A'B'C'$; and if A, B, C be the poles of the circles, show that the

$$\text{area } A'B'C' = (A \cos \alpha + B \cos \beta + C \cos \gamma - \pi) r^2.$$

[The area of $AB'C'$ is to the area of the segment on which it is situated as $A : 2\pi$; therefore

$$AB'C' = Ar^2 (1 - \cos \alpha).$$

Similarly,

$$BC'A' = Br^2 (1 - \cos \beta), \quad \text{and} \quad CA'B' = Cr^2 (1 - \cos \gamma);$$

therefore, &c.].

19. If two sides of a spherical triangle be given, and its area a maximum, the arcs drawn perpendicular to the sides at their middle points pass through the middle point of the base.

[For the middle point of the base is the pole of the circumcircle.]

20. If the area of a spherical triangle be one-fourth of the area of the sphere, the arcs joining the middle points of its sides are quadrants.

(*London University.*)

[The triangle DPQ (fig. 38), shows that

$$\cos \tfrac{1}{2} E = \frac{\cos \gamma}{\cos \frac{1}{2} c} = \frac{\cos \beta}{\cos \frac{1}{2} b} = \frac{\cos \alpha}{\cos \frac{1}{2} a}.\Big]$$

21. If the area of a spherical triangle be one-fourth the area of the sphere, show that the bisector of a side is the supplement of half that side.

[The middle points of its sides are the circumcentres of the colunar triangles, since in each of them we have, by the given condition, the vertical angle equal to the sum of the base angles.]

22. Given the base and area, show that the arc joining the middle points of the sides is constant; and if it is a quadrant, then the area of the triangle is πr^2. (See fig. 38.)

23. What relation connects the area of a triangle with the sum of the sides of its polar triangle?

24. Two circles of angular radii, α and β, intersect orthogonally on a sphere of radius r; find in any manner the area common to the two. (*London University.*)

[Let A and B be the angles subtended at the poles of the circles by their common chord, M and N their common points, δ the distance between their poles; then, if x be the common area, we have

$$\text{sector } AMN + \text{sector } BMN = \text{quadrilateral } AMBN + x;$$

therefore

$$x + (A + B - \pi)\, r^2 = Ar^2 (1 - \cos\alpha) + Br^2 (1 - \cos\beta);$$

or

$$x = (\pi - A\cos\alpha - B\cos\beta)\, r^2.$$

A, B are determined from the equations

$$\sin\alpha = \sin\delta \sin\tfrac{1}{2}B,\quad \sin\beta = \sin\delta\sin\tfrac{1}{2}A,\quad \cos\delta = \cos\alpha\cos\beta.]$$

25. Find the surface of an equilateral and equiangular polygon, and the value of each of the angles when its area is half the surface of the sphere. (*Cambridge Exam. Papers.*)

26. If E be the spherical excess of a triangle; prove that

$$\tfrac{1}{2}E = \tan\tfrac{1}{2}a \tan\tfrac{1}{2}b \sin C - \tfrac{1}{2}(\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b)^2 \sin 2C + \&c.$$

[We have
$$\tan\tfrac{1}{2}E = \frac{\tan\frac{1}{2}a\tan\frac{1}{2}b\sin C}{1 + \tan\frac{1}{2}a\tan\frac{1}{2}b\cos C};$$

and if we have
$$\tan x = \frac{\lambda\sin\alpha}{1 + \lambda\cos\alpha},$$

then
$$x = n\pi + \lambda\sin\alpha - \tfrac{1}{2}\lambda^2\sin 2\alpha + \tfrac{1}{3}\lambda^3\sin 3\alpha - \ldots\ldots$$

See Todhunter's *Plane Trig.*, Art. 298.]

CHAPTER VIII.

THE CHORDAL TRIANGLE.

116. Frequent reference has already been made to the chordal triangle, or the triangle formed by the chords of the sides of a spherical triangle. We shall now point out some of the relations which exist between the parts of a spherical triangle and its chordal triangle.

We shall denote the angles of the chordal triangle by A', B', C', its sides by a', b', c'; and if r be the radius of the sphere, we have

$$a' = 2r \sin \tfrac{1}{2}a, \quad b' = 2r \sin \tfrac{1}{2}b, \quad c' = 2r \sin \tfrac{1}{2}c.$$

117. **Problem.**—*Having given the sides of a spherical triangle, to determine the angles of the chordal triangle.*

By Plane Trigonometry,

$$a'^2 = b'^2 + c'^2 - 2b'c' \cos A'.$$

Dividing both sides of this equation by $4r^2$, we find

$$\sin^2 \tfrac{1}{2}a = \sin^2 \tfrac{1}{2}b + \sin^2 \tfrac{1}{2}c - 2 \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c \cos A'.$$

Hence

$$\cos A' = \frac{1 + \cos a - \cos b - \cos c}{4 \sin \frac{1}{2}b \sin \frac{1}{2}c},$$

or,

with similar values for $\cos B'$ and $\cos C'$.

Referring to the third of the Miscellaneous Examples of Chap. III., it will be seen that this expression for $\cos A'$ is identical with that for the cosine of the arc joining the middle points of the sides of the colunar triangle on the side a, and we may conclude, therefore, that the angles of the chordal triangle are respectively equal to the arcs joining the middle points of the sides of the colunars on the sides opposite to them—a relation which may be proved geometrically, as follows:—

118. *The angles of the chordal triangle are, respectively, equal to the arcs joining the middle points of the sides of the colunar triangles.*

Let ABC (fig. 43) be a spherical triangle, O the centre of the sphere, and C' the point diametrically opposite to C; M and N the middle points of $C'A$ and $C'B$. Then

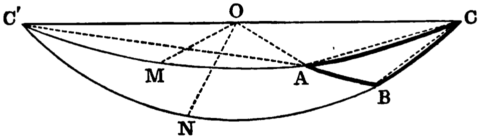

Fig. 43.

OM is parallel to AC, since they are both perpendicular to AC'. Similarly, ON is parallel to BC. Therefore the angle MON or the arc MN is equal to the angle C' of the chordal triangle.

Referring to Fig. 10, Art. 20, it will be seen that O is the pole of the circumcircle of the colunar on AB, and that

$$AOB = LM = 2XY;$$

that is, the angle which any side of a triangle subtends at the pole of its circumcircle is double of the arc joining the middle points of the sides of the corresponding colunar, or (by the foregoing) double of the corresponding angle of the chordal triangle.

This may be proved independently, as follows :—

119. *The angle subtended by a side of a triangle at the pole of its circumcircle is double of the corresponding angle of the chordal triangle.*

Let H be the pole of the circumcircle; then the planes of the great circles AH and BH are each perpendicular to the plane of the circumcircle; and therefore the angle between them is equal to the angle which the side AB of the chordal triangle subtends at the *centre* of the circumcircle; that is, is double of the corresponding angle of the chordal triangle.

Cor. 1.—When $C = A + B$, the chordal triangle is right-angled at C. For AB passes through the pole of the circumcircle, and therefore subtends an angle π at it. Hence $C' = \frac{1}{2}\pi$, as is obvious, since A and B are diametrically opposite points on the circumcircle.

Cor. 2.—

$$\cos A' = \cos \tfrac{1}{2} a \sin (S - A),$$

$$\cos B' = \cos \tfrac{1}{2} b \sin (S - B),$$

$$\cos C' = \cos \tfrac{1}{2} c \sin (S - C).$$

[These values follow at once by dropping perpendiculars from H on the sides of the triangle.]

120. **Problem.**—*Having given two sides and the included angle of a spherical triangle, to find the corresponding angle of the chordal triangle.*

Since C' is equal to the arc MN, joining the middle points of the sides of the colunar, we have at once

$$\cos C' = \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b + \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos C;$$

with similar values for $\cos A'$ and $\cos B'$.

No difficulty will be found in identifying these values of $\cos A'$, $\cos B'$, $\cos C'$ with those given in Art. 117.

121. *If the sides of a spherical triangle be small compared with the radius of the sphere, to find the excess of an angle of the spherical triangle over the corresponding angle of the chordal triangle.*

Let θ denote the excess; then

$$\cos A' = \cos (A - \theta) = \cos A + \theta \sin A, \text{ nearly.}$$

But by Art. 120,

$$\cos A' = \sin^2 \tfrac{1}{4} (b + c) - \sin^2 \tfrac{1}{4} (b - c)$$
$$+ \left[1 - \sin^2 \tfrac{1}{4} (b + c) - \sin^2 \tfrac{1}{4} (b - c)\right] \cos A.$$

Therefore

$$\theta \sin A = \sin^2 \tfrac{1}{4} (b + c) (1 - \cos A) - \sin^2 \tfrac{1}{4} (b - c) (1 + \cos A);$$

from which we find

$$\theta = \tan \tfrac{1}{2} A \sin^2 \tfrac{1}{4} (b + c) - \cot \tfrac{1}{2} A \sin^2 \tfrac{1}{4} (b - c).$$

This expression gives the *circular measure* of the excess as a function of two sides and the included angle. The value in seconds is obtained by dividing by the circular measure of one second.

Cor.—

$$\theta \sin A = \cos \tfrac{1}{2} a \sin (S - A) - \cos A.$$

[For $\cos A' = \cos A + \theta \sin A = \cos \tfrac{1}{2} a \sin (S - A)$.

(Art. 119, *Cor.* 2).]

Examples.

1. If A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3; be the angles of the chordal triangles of the colunars, prove that

$\cos A_1 = \cos \tfrac{1}{2} a \sin S$, $\cos B_1 = \sin \tfrac{1}{2} b \sin (S - C)$, $\cos C_1 = \sin \tfrac{1}{2} c \sin (S - B)$,

$\cos A_2 = \sin \tfrac{1}{2} a \sin (S - C)$, $\cos B_2 = \cos \tfrac{1}{2} b \sin S$, $\cos C_2 = \sin \tfrac{1}{2} c \sin (S - A)$,

$\cos A_3 = \sin \tfrac{1}{2} a \sin (S - B)$, $\cos B_3 = \sin \tfrac{1}{2} b \sin (S - A)$, $\cos C_3 = \cos \tfrac{1}{2} c \sin S$.

2. Prove that

$$\frac{\sin(A'-B')}{\sin C'}=\frac{\sin(A-B)}{\sin C}.$$

(*London Univ. Exam. Papers.*)

3. Using the notation of Ex. 1, show that

$$\cos A_1 : \cos B_2 : \cos C_3 = \cos \tfrac{1}{2}a : \cos \tfrac{1}{2}b : \cos \tfrac{1}{2}c.$$

4. If one of the arcs joining the middle points of the sides of a triangle is a quadrant, the others are quadrants; and hence, if the chordal triangle of one of the colunars is right-angled, the chordal triangles of the other colunars are right-angled, and the middle points of the sides of the original triangle are the poles of the circumcircles of the colunars.

5. Show, by Ex. 4, that if one of the colunar triangles is the greatest that can be constructed with its two sides, the other colunars are also of maximum area.

6. If $C = A + B$, prove that

$$\cos C = -\tan \tfrac{1}{2}a \tan \tfrac{1}{2}b.$$

[The chordal triangle is right-angled at C.]

7. If two sides a, b of a triangle be given and the area a maximum, show that the arc joining the middle points of the sides is given by the equation

$$\cos\gamma=\frac{\cos a+\cos b}{\cos\frac{1}{2}a\cos\frac{1}{2}b}.$$

8. In the same case, show that the third side, c, is found from the equation

$$2\cos^2\tfrac{1}{2}c=\cos a+\cos b.$$

9. Show that the sum of the three arcs joining the middle points of the sides of the colunars is equal to two right angles, the sides of the original triangle being regarded as the bases of the colunars.

[They are equal to A', B', C', respectively.]

10. Prove that

$$\cos^2\tfrac{1}{2}a\sin^2 S+\sin^2\tfrac{1}{2}b\sin^2(S-C)+\sin^2\tfrac{1}{2}c\sin^2(S-B)$$
$$+\,2\cos\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c\sin S\sin(S-B)\sin(S-C)=1.$$

[See Ex. 1, and remember that

$$1-\cos^2 A'-\cos^2 B'-\cos^2 C'-2\cos A'\cos B'\cos C'=0].$$

11. Prove that

$$\sum\left(\frac{1+\cos a-\cos b-\cos c}{\sin\frac{1}{2}b\sin\frac{1}{2}c}\right)^2+\frac{1}{2}\left(\frac{1+\cos a-\cos b-\cos c}{\sin\frac{1}{2}b\sin\frac{1}{2}c}\right)(\ldots)(\ldots)=16.$$

(*Educational Times.*)

12. Prove that

$$\sin^2(S-A)\cos^2\tfrac{1}{2}a+\sin^2(S-B)\cos^2\tfrac{1}{2}b+\sin^2(S-C)\cos^2\tfrac{1}{2}c$$
$$+2\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\sin(S-A)\sin(S-B)\sin(S-C)=1.$$

13. Prove that

(α) $\sin\frac{1}{2}b\cos A'+\sin\frac{1}{2}a\cos B'=\sin\frac{1}{2}c.$ (Art 117.)

(β) $\sin\frac{1}{2}a\cos\frac{1}{2}b\sin(S-B)+\sin\frac{1}{2}b\cos\frac{1}{2}a\sin(S-A)=\sin\frac{1}{2}c.$

14. What is the *analogue in plano* to Ex. 13 (β)?

Ans. $c=b\cos A+a\cos B.$

15. Having given the base and the arc joining the middle points of the colunar on the base, the circumcircle is fixed.

[For the vertical angle of the chordal triangle is given.]

16. Prove the equation

$$\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c\sin(S-A)+\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\sin S=\cos\tfrac{1}{2}a.$$

[$\sin\frac{1}{2}c\cos B_3+\cos\frac{1}{2}b\cos C_3=\cos\frac{1}{2}a$ (Ex. 1).]

17. Prove the relation

$$8\cos A'\cos B'\cos C'=\frac{\sin(B+C-A)\sin(C+A-B)\sin(A+B-C)}{\sin A\sin B\sin C}.$$

18. If $A+B+C=2\pi$, prove that

(α) $\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c=1.$

(β) $\cos C=-\cot\frac{1}{2}a\cot\frac{1}{2}b.$ (Cf. Ex. 6.)

19. Solve the equations

$$\sin b\cos c\sin Z+\sin c\cos b\sin Y=\sin a,$$
$$\sin c\cos a\sin X+\sin a\cos c\sin Z=\sin b,$$
$$\sin a\cos b\sin Y+\sin b\cos a\sin X=\sin c.$$

[Let $2a$, $2b$, $2c$ be the sides of a spherical triangle, then

$$\sin X = \frac{1 + \cos 2a - \cos 2b - \cos 2c}{4 \sin b \sin c},$$

with similar values for Y and Z.]

20. Show, by applying Art. 119, *Cor.* 2, to the colunar triangle, that

$$\frac{\cos \alpha}{\cos \frac{1}{2} a} = \frac{\cos \beta}{\cos \frac{1}{2} b} = \frac{\cos \gamma}{\cos \frac{1}{2} c} = \sin S,$$

where α, β, γ are the arcs joining the middle points of the sides.

21. What is the geometrical interpretation of Ex. 17?

Ans. If one angle of a spherical triangle be equal to the sum of the other two, the chordal triangle is right-angled.

CHAPTER IX.

CONCURRENT ARCS AND CONCYCLIC POINTS.

122. Condition that Three Arcs through the Vertices of a Triangle should be Concurrent.

If three arcs, OA, OB, OC, drawn from a point O to the vertices of a triangle, meet the opposite sides in points A′, B′, C′, and divide them into segments a_1, a_2; b_1, b_2; c_1, c_2, *respectively, we have the relation*

$$\frac{\sin a_1}{\sin a_2} \cdot \frac{\sin b_1}{\sin b_2} \cdot \frac{\sin c_1}{\sin c_2} = 1;$$

and conversely—*if points A′, B′, C′ divide the sides of a triangle into segments connected by the above relation, the arcs AA′, BB′, CC′ will meet in a point.**

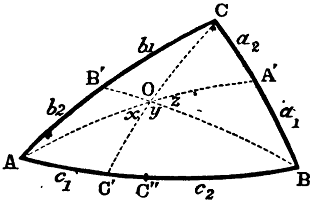

Fig. 44.

Let *AA′, BB′, CC′* (fig. 44) meet at *O* and divide the sides into segments, as marked in figure.

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* N.B.—In all cases in which we speak of any number of arcs passing through a point, it is to be understood that they pass through the diametrically opposite point also.

The triangles $BA'O$ and $CA'O$ give us

$$\sin a_1 \sin A' = \sin BO \sin z,$$

and
$$\sin a_2 \sin A' = \sin CO \sin x.$$

Therefore
$$\frac{\sin a_1}{\sin a_2} = \frac{\sin BO \sin z}{\sin CO \sin x}.$$

Similarly,
$$\frac{\sin b_1}{\sin b_2} = \frac{\sin CO \sin y}{\sin AO \sin z},$$

and
$$\frac{\sin c_1}{\sin c_2} = \frac{\sin AO \sin x}{\sin BO \sin y}.$$

Multiplying these three results together, we obtain the relation in question.

Again, *conversely*, if this relation holds among the segments of the sides, the arcs AA', BB', CC' will pass through a point. For, if possible, let A', B', C'' be three points dividing the sides into segments, which satisfy the above relation, and suppose the arc CO not to pass through C''. Then, since AA', BB', CC' are concurrent, we have

$$\frac{\sin a_1}{\sin a_2} \cdot \frac{\sin b_1}{\sin b_2} \cdot \frac{\sin c_1}{\sin c_2} = 1.$$

Also, by hypothesis,

$$\frac{\sin a_1}{\sin a_2} \cdot \frac{\sin b_1}{\sin b_2} \cdot \frac{\sin AC''}{\sin BC''} = 1.$$

Therefore

$$\frac{\sin c_1}{\sin c_2} = \frac{\sin AC''}{\sin BC''},$$

which is absurd, since the arc AB is less than 180°.

123. If the arcs AA', BB', CC' divide the angles A, B, C into segments A_1, A_2; B_1, B_2; C_1, C_2; opposite the corresponding segments of the sides, then we have

$$\frac{\sin a_1}{\sin a_2} = \frac{\sin c \sin A_1}{\sin b \sin A_2}, \text{ \&c., \&c.,}$$

and therefore

$$\frac{\sin A_1}{\sin A_2} \cdot \frac{\sin B_1}{\sin B_2} \cdot \frac{\sin C_1}{\sin C_2} = 1.$$

And just as before, it holds conversely, that if this relation exists among the segments into which three arcs divide the angles of a triangle, then these arcs, produced if necessary, will pass through a point. This then may also be regarded as a criterion of the concurrency of three arcs.

Hence we have the following theorem:—*If three arcs passing through the vertices of a triangle be concurrent, three arcs equally inclined to the bisectors of the angles of the triangle will also be concurrent.*

For, if the former triad divide the angles into segments A_1, A_2; B_1, B_2; C_1, C_2; the latter triad will divide them into segments A_2, A_1; B_2, B_1; C_2, C_1; and, therefore, if the former triad meet in a point, so also will the latter.

If this latter triad make segments a_1', a_2', b_1', b_2', c_1', c_2', on the sides, we have, as at the beginning of this Article,

$$\frac{\sin a_1'}{\sin a_2'} = \frac{\sin c \,.\, \sin A_2}{\sin b \,.\, \sin A_1}. \quad \text{But} \quad \frac{\sin A_2}{\sin A_1} = \frac{\sin a_2 \,.\, \sin c}{\sin a_1 \,.\, \sin b}.$$

$$\left.\begin{aligned}
&\text{Therefore} \qquad \frac{\sin a_1'}{\sin a_2'} = \frac{\sin a_2}{\sin a_1} \cdot \frac{\sin^2 c}{\sin^2 b}.\\
&\text{Similarly,} \qquad \frac{\sin b_1'}{\sin b_2'} = \frac{\sin b_2}{\sin b_1} \cdot \frac{\sin^2 a}{\sin^2 c};\\
&\text{and} \qquad \frac{\sin c_1'}{\sin c_2'} = \frac{\sin c_2}{\sin c_1} \cdot \frac{\sin^2 b}{\sin^2 a}.
\end{aligned}\right\}$$

These expressions also show that if one triad be concurrent, the other triad will also be concurrent.

Examples.

1. Prove that the following arcs are concurrent:—

(α) The medians or bisectors of sides.

(β) The symmedians.

$$\left[\text{In } (\alpha) \text{ we have } \frac{\sin a_1}{\sin a_2} = 1, \text{ \&c.; and in } (\beta) \text{ we have } \frac{\sin a_1'}{\sin a_2'} = \frac{\sin^2 c}{\sin^2 b}, \text{ \&c.}\right]$$

2. Show that—

(α) The three internal bisectors of the angles;

(β) Two external and one internal;

are concurrent.

[$\sin a_1 : \sin a_2 :: \sin c : \sin b$, or apply Art. 123.]

3. The perpendiculars of a spherical triangle are concurrent. (See Art. 54).

$$\left[\frac{\sin c_1}{\sin c_2} = \frac{\sin b \cos A}{\sin a \cos B}, \text{ \&c.}\right].$$

4. Show the concurrency of the arcs joining the vertices of a triangle to—

(α) The points of contact of the sides with the in-circle;

(β) The points of contact of the sides with each of the ex-circles;

[$a_2 = b_1$, $b_2 = c_1$, $c_2 = a_1$. Therefore, &c.].

5. Prove that the three arcs drawn through the vertices of a triangle, bisecting its area, are concurrent.

[We have $\dfrac{\sin \frac{1}{2} a_1}{\sin \frac{1}{2} a_2} = \dfrac{\cos \frac{1}{2} c}{\cos \frac{1}{2} b}$, &c.; hence, if we calculate $\sin a_1$, $\sin a_2$, &c., we find the criterion satisfied. For a simple geometrical proof see Art. 157.]

124. **Condition that Three Points on the Sides of a Triangle should be Concyclic.**—*If a great circle meet the sides of a triangle in points A', B', C', dividing them into segments a_1, a_2; b_1, b_2; c_1, c_2, we have the relation*

$$\frac{\sin a_1}{\sin a_2} \cdot \frac{\sin b_1}{\sin b_2} \cdot \frac{\sin c_1}{\sin c_2} = -1;\text{*}$$

---

* The negative sign is taken, because any arc drawn across the sides of a triangle must cut either *one* or *three* of the sides externally.

and conversely—*If three points be taken on the sides of a triangle, and if this relation holds among the segments into which they divide the sides, then the points will lie on the same great circle.*

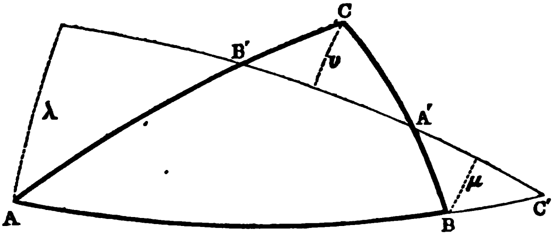

Fig. 45.

Let λ, μ, ν be the perpendiculars from ABC on the great circle $A'B'C'$. Then, if $AC' = c_1$, and $C'B = c_2$, as before, we have

$$\frac{\sin a_1}{\sin a_2} = \frac{\sin \mu}{\sin \nu}.$$

Similarly,
$$\frac{\sin b_1}{\sin b_2} = \frac{\sin \nu}{\sin \lambda},$$

and
$$\frac{\sin c_1}{\sin c_2} = -\frac{\sin \lambda}{\sin \mu}.$$

Therefore
$$\frac{\sin a_1}{\sin a_2} \cdot \frac{\sin b_1}{\sin b_2} \cdot \frac{\sin c_1}{\sin c_2} = -1.$$

The converse—viz., if the segments a_1, a_2; b_1, b_2; c_1, c_2 are connected by the above relation, the points A', B', C' lie on the same great circle—follows exactly as in the foregoing Article, by supposing them not to lie on the same great circle.

Remark.—The relation of Art. 55 (1) connecting the mutual distances of three points with the perpendiculars from them on any fixed great circle may also be used with advantage to determine if three points lie on the same great circle.

125. **Theorem.**—*If three great circles, drawn from the vertices of a triangle, pass through a point O, and meet the opposite sides in points A', B', C', the sides of the triangle A', B', C' will meet the corresponding sides of the triangle ABC in points which lie on a great circle.*

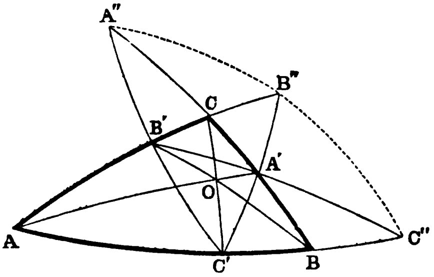

Fig. 46.

Since AA', BB', CC' (fig. 46) are concurrent, we have

$$\frac{\sin a_1}{\sin a_2} \cdot \frac{\sin b_1}{\sin b_2} \cdot \frac{\sin c_1}{\sin c_2} = 1 ;$$

and if $A'B'$ meet AB in a point C'', we have, since B', A', C'' are on the same great circle,

$$\frac{\sin a_1}{\sin a_2} \cdot \frac{\sin b_1}{\sin b_2} \cdot \frac{\sin AC''}{\sin BC''} = -1.$$

Hence $$\frac{\sin AC''}{\sin BC''} = -\frac{\sin c_1}{\sin c_2}.$$

Similarly, $$\frac{\sin BA''}{\sin CA''} = -\frac{\sin a_1}{\sin a_2},$$

and $$\frac{\sin CB''}{\sin AB''} = -\frac{\sin b_1}{\sin b_2}.$$

Therefore $$\frac{\sin AC''}{\sin BC''}\cdot\frac{\sin BA''}{\sin CA''}\cdot\frac{\sin CB''}{\sin AB''} = -1;$$

and hence the points A'', B'', C'' lie on a great circle.

Cor.—If A' and B' be the middle points of the sides a and b, then C' will be the middle point of the side c. But it has been proved that the arc joining the middle points of two sides meets the third side at 90° from its middle point (Art. 20); hence, if $C'C''$ and $A'A''$ are quadrants, then $B'B''$ will also be a quadrant.

Therefore, since A'', B'', C'' are concyclic, by considering the quadrilateral $C'A'C''A''$, it follows that, *if two diagonals, $A'A''$ and $C'C''$, of a quadrilateral be quadrants, the third diagonal, $B'B''$, will also be a quadrant.*

Examples.

1. The arcs joining the middle points of the sides of a triangle meet the opposite sides in points which lie on a great circle, having the pole of the circumcircle for its pole.

[See fig (10) and note that the points E and D are 90° distant from Z.]

2. The great circles joining the feet of the perpendiculars of a triangle, in pairs, meet the opposite sides in points which lie on a great circle.

3. The great circles joining in pairs the points of contact of the in-circle, or of any one of the ex-circles, with the sides, meet the opposite sides in points which lie on a great circle.

4. The arcs joining in pairs the points where the bisectors of the angles of a triangle meet the opposite sides, intersect the opposite sides in points which lie on a great circle.

5. The medians of a triangle intersect at O; prove that

$$\sin OA : \sin OA' :: 2\cos\tfrac{1}{2}a : 1;$$

$$\sin OB : \sin OB' :: 2\cos\tfrac{1}{2}b : 1;$$

$$\sin OC : \sin OC' :: 2\cos\tfrac{1}{2}c : 1.$$

[Consider the triangle ACC' and the transversal BB' drawn across it, and apply the relation of Art. 124.]

6. If arcs AA', BB', CC' be drawn through the vertices of a triangle passing through a point O, and cutting the opposite sides in A', B', C', prove that

$$(\alpha)\quad \sum \frac{\sin OA' \cos OA}{\sin AA'} = 1.$$

$$(\beta)\quad \sum \frac{\tan OA'}{\tan OA' + \tan OA} = 1.$$

$$(\gamma)\quad \sum \frac{\sin OA \, . \cos OA'}{\sin AA'} = 2.$$

$$(\delta)\quad \sum \frac{\tan OA}{\tan OA + \tan OA'} = 2.$$

7. If a triangle be inscribed in a fixed small circle, and if two of its sides touch two fixed spherical curves, its third side will envelop a third spherical curve, such that the vertices of the triangle connect concurrently with the points of contact of the opposite sides with their envelopes.

[Take a consecutive position of the triangle, and it will be seen that the segments of the sides, made by the points of contact, satisfy the relation of Art. 122.]

CHAPTER X.

DIRECTION ANGLES AND DIRECTION COSINES.

126. **Tri-quadrantal Triangle.**—If each side of a spherical triangle be a quadrant, each vertex will be the pole of the opposite side, and each angle a right angle. Such a triangle we shall refer to as being *Tri-quadrantal.* It may be observed that the radii from the centre of the sphere to the vertices of a tri-quadrantal triangle are mutually perpendicular to each other, the angle between two radii being equal to the arc joining their extremities; so also the planes of the great circles forming the triangle are mutually perpendicular.

127. **Direction Cosines.**—The angular distances of a point on the surface of a sphere, from the vertices of a fixed tri-quadrantal triangle on it, are called the *Direction Angles* of the point, and the cosines of these angles are called the *Direction Cosines* of the point. Thus, if P (fig. 47) be any point on the surface of a sphere, ABC a fixed tri-quadrantal triangle, the arcs PA, PB, PC are the direction angles of P. These we shall denote by α, β, γ, respectively. It is obvious that α, β, γ are the angles which the radius to P makes with the radii to A, B, C, and if any two of these be known, the *direction* of the radius to P is determined; thus, any one of the direction cosines

of a point is determined by the other two, and therefore a relation exists among them. This relation we proceed to determine.

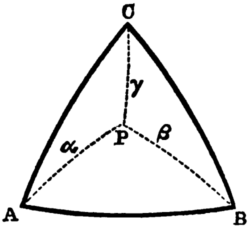

Fig. 47.

128. Relation connecting the Direction Cosines of a Point.—*The sum of the squares of the direction cosines of any point is equal to unity.*

Let α, β, γ be the direction angles of any point P (fig. 47). Since the triangle APC is quadrantal, we have

$$\cos\alpha = \sin\gamma \cos ACP,$$

and $\cos\beta = \sin\gamma \cos BCP$ (from triangle BCP).

But ACP and BCP are complementary; hence, squaring and adding the above equations, we have

$$\cos^2\alpha + \cos^2\beta = \sin^2\gamma.$$

Therefore $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$[*]

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[*] This relation merely asserts that the square of the diagonal of a rectangular parallelopiped is equal to the sum of the squares of its three edges; the diagonal in this case being the radius to P, and the edges the perpendiculars from P on the planes of the sides of the triangle ABC. The expression given in Art. 129 for $\cos\delta$ may be easily deduced from the same principle.

Cor.—If x, y, z be the perpendiculars from any point on the sides of a tri-quadrantal triangle; then

$$\sin^2 x + \sin^2 y + \sin^2 z = 1.$$

129. **Angular Distance between two Points.**—*Having given the direction angles of two points, to find the angular distance between them.*

Let α, β, γ; α', β', γ' be the direction angles of two points P, P' (fig. 48), and δ the angular distance between them.

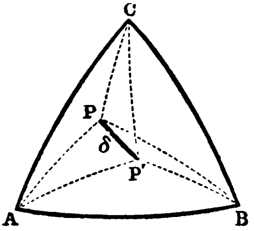

Fig. 48.

From the triangle PAP' we have

$$\cos\delta = \cos\alpha\cos\alpha' + \sin\alpha\sin\alpha'\cos PAP'. \qquad (1)$$

But $\cos PAP' = \cos(PAB - P'AB)$.

Now $\cos\beta = \sin\alpha\cos PAB$, $\cos\beta' = \sin\alpha'\cos P'AB$;

also $\cos\gamma = \sin\alpha\sin PAB$, since $PAB = 90° - PAC$;

and $\cos\gamma' = \sin\alpha'\sin P'AB$, since $P'AB = 90° - P'AC$.

From these equations we find

$$\cos PAP' = \frac{\cos\beta\cos\beta' + \cos\gamma\cos\gamma'}{\sin\alpha\sin\alpha'};$$

which, when substituted in (1), gives us

$$\cos\delta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'. \qquad (2)$$

In this equation δ is the angle between the radii to P and P', and α, β, γ; α', β', γ' the angles these radii make with the radii to A, B, C. *Hence the above equation gives the angle between any two lines in terms of the angles they make with three fixed rectangular axes.*

Cor.—If x, y, z; x', y', z' be the perpendiculars from any two points on the sides of a tri-quadrantal triangle, δ, the angular distance between them is given by the equation

$$\cos\delta = \sin x \sin x' + \sin y \sin y' + \sin z \sin z'.$$

Example.

If the angular distances of a point, P, from the vertices A, B, C, of a tri-quadrantal triangle be ξ, η, ζ, and if the direction angles of P, A, B, C, referred to any other tri-quadrantal triangle, be α, β, γ; $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; $\alpha_3, \beta_3, \gamma_3$, respectively; prove that

$$\cos\xi = \cos\alpha \cos\alpha_1 + \cos\beta \cos\beta_1 + \cos\gamma \cos\gamma_1.$$
$$\cos\eta = \cos\alpha \cos\alpha_2 + \cos\beta \cos\beta_2 + \cos\gamma \cos\gamma_2.$$
$$\cos\zeta = \cos\alpha \cos\alpha_3 + \cos\beta \cos\beta_3 + \cos\gamma \cos\gamma_3.$$

$$\cos\alpha = \cos\xi \cos\alpha_1 + \cos\eta \cos\alpha_2 + \cos\zeta \cos\alpha_3.$$
$$\cos\beta = \cos\xi \cos\beta_1 + \cos\eta \cos\beta_2 + \cos\zeta \cos\beta_3.$$
$$\cos\gamma = \cos\xi \cos\gamma_1 + \cos\eta \cos\gamma_2 + \cos\zeta \cos\gamma_3.$$

These equations enable us to determine the direction angles of any point with reference to one triangle when its direction angles, with reference to the other, are known, the relative position of one triangle with respect to the other being also known.

130. **Expression for** sin δ.—*To find the sine of the angle between any two lines in terms of the angles they make with three fixed rectangular axes.*

By the foregoing Article we have

$$\cos^2\delta = \Sigma \cos^2\alpha \cos^2\alpha' + 2\Sigma \cos\beta \cos\beta' \cos\gamma \cos\gamma'.$$

Using the relation

$$\cos^2 \alpha' + \cos^2 \beta' + \cos^2 \gamma' = 1,$$

this transforms into

$$\begin{aligned}\cos^2 \delta = \Sigma \cos^2 \alpha\, (1 - \cos^2 \beta' - \cos^2 \gamma') \\ + 2\Sigma \cos\beta \cos\beta' \cos\gamma \cos\gamma'\end{aligned}$$

$$= \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma - \Sigma\, (\cos\beta \cos\gamma' - \cos\beta' \cos\gamma)^2$$
$$= 1 - \Sigma\, (\cos\beta \cos\gamma' - \cos\beta' \cos\gamma)^2.$$

Therefore

$$\sin^2 \delta = \Sigma\, (\cos\beta \cos\gamma' - \cos\beta' \cos\gamma)^2.$$

This expression may also be obtained directly from the identical relation

$$\begin{aligned}(bc' - b'c)^2 + (ca' - c'a)^2 + (ab' - a'b)^2 \\ = (a^2 + b^2 + c^2)\,(a'^2 + b'^2 + c'^2) - (aa' + bb' + cc')^2 ;\end{aligned}$$

obtained by squaring the array

$$\left.\begin{matrix} a & b & c \\ a' & b' & c' \end{matrix}\right\}.$$

131. **Direction Cosines of the Pole of the Arc joining two Points.**—Let α_1, β_1, γ_1, α_2, β_2, γ_2 be the direction angles of two points; α, β, γ the direction angles of the pole of the arc joining them. Then since the arcs joining $\alpha\beta\gamma$ to the other points are each 90°, we have, by Art. 129,

$$\cos\alpha \cos\alpha_1 + \cos\beta \cos\beta_1 + \cos\gamma \cos\gamma_1 = 0 ;$$

and

$$\cos\alpha \cos\alpha_2 + \cos\beta \cos\beta_2 + \cos\gamma \cos\gamma_2 = 0.$$

Hence

$$\frac{\cos\alpha}{\cos\beta_1\cos\gamma_2-\cos\gamma_1\cos\beta_2}=\frac{\cos\beta}{\cos\gamma_1\cos\alpha_2-\cos\alpha_1\cos\gamma_2}$$

$$=\frac{\cos\gamma}{\cos\alpha_1\cos\beta_2-\cos\beta_1\cos\alpha_2}$$

$$=\sqrt{\frac{\cos^2\alpha+\cos^2\beta+\cos^2\gamma}{\Sigma\,(\cos\beta_1\cos\gamma_2-\cos\gamma_1\cos\beta_2)^2}}=\frac{1}{\sin\delta},$$

where δ is the angular distance between the given points.

Therefore

$$\cos\alpha\sin\delta=\cos\beta_1\cos\gamma_2-\cos\gamma_1\cos\beta_2,$$

$$\cos\beta\sin\delta=\cos\gamma_1\cos\alpha_2-\cos\alpha_1\cos\gamma_2,$$

$$\cos\gamma\sin\delta=\cos\alpha_1\cos\beta_2-\cos\beta_1\cos\alpha_2.$$

132. **Relation among the Six Arcs of Connection of Four Points on a Sphere.**—Let the direction angles of the points 1, 2, 3, 4, be α_1, β_1, γ_1; α_2, β_2, γ_2, &c. Then if we perform the multiplication

$$\begin{vmatrix}\cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 & 0\\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 & 0\\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 & 0\\ \cos\alpha_4 & \cos\beta_4 & \cos\gamma_4 & 0\end{vmatrix}\times\begin{vmatrix}\cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 & 0\\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 & 0\\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 & 0\\ \cos\alpha_4 & \cos\beta_4 & \cos\gamma_4 & 0\end{vmatrix},$$

the product will obviously be zero, since each of these determinants is zero; and observing that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,$$

and $\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=\cos\overline{12},$

where $\overline{12}$ represents the arc joining the points 1 and 2, we have

$$\begin{vmatrix} 1 & \cos\overline{12} & \cos\overline{13} & \cos\overline{14} \\ \cos\overline{21} & 1 & \cos\overline{23} & \cos\overline{24} \\ \cos\overline{31} & \cos\overline{32} & 1 & \cos\overline{34} \\ \cos\overline{41} & \cos\overline{42} & \cos\overline{43} & 1 \end{vmatrix} = 0,$$

which is the required relation in a determinant form.

It is clear from what has been said in Art. 129, that this is the relation connecting the angles which four lines in space make with three fixed rectangular axes, the lines being parallel to the radii to the points 1, 2, 3, 4. This relation may be obtained directly, though not in a determinant form, by taking the point 4 inside the triangle formed by the points 1, 2, 3, and observing that if the connectors of 1, 2, 3 subtend angles A, B, C at 4, then $A + B + C = 2\pi$, and

$$1 - \cos^2 A - \cos^2 B - \cos^2 C + 2\cos A \cos B \cos C = 0.$$

Substituting in this equation the values of $\cos A$, &c., in terms of the arcs joining the points, and reducing, we obtain

$$a^2 + b^2 + c^2 + \alpha^2 + \beta^2 + \gamma^2 - a^2\alpha^2 - b^2\beta^2 - c^2\gamma^2 + 2bc\beta\gamma + 2ca\gamma\alpha$$
$$+ 2ab\alpha\beta - 2abc - 2\alpha\beta\gamma - 2b\gamma\alpha - 2c\alpha\beta = 1,$$

where a, b, c; α, β, γ are taken, for the sake of brevity, to represent the cosines of the six arcs in question.

Examples.

1. To find a relation connecting the sides of a triangle with the perpendiculars x, y, z, let fall on them from any point on the sphere.

[If α, β, γ be the angular distances of any point P on the sphere from the vertices of a triangle ABC, the foregoing determinant becomes

$$\begin{vmatrix} 1 & \cos c & \cos b & \cos\alpha \\ \cos c & 1 & \cos a & \cos\beta \\ \cos b & \cos a & 1 & \cos\gamma \\ \cos\alpha & \cos\beta & \cos\gamma & 1 \end{vmatrix} = 0.$$

This applied to the polar triangle, the sides of which are $\pi - A$, $\pi - B$, $\pi - C$, becomes

$$\begin{vmatrix} -1 & \cos C & \cos B & \sin x \\ \cos C & -1 & \cos A & \sin y \\ \cos B & \cos A & -1 & \sin z \\ \sin x & \sin y & \sin z & -1 \end{vmatrix} = 0,$$

since the cosine of the angular distance of any point from a vertex of the polar triangle is equal to the sine of the perpendicular from that point on the corresponding side of the original triangle.

The coefficient of $\sin^2 x$ in this determinant is $\sin^2 A$, and the coefficient of $\sin y \sin z$ is $2(\cos A + \cos B \cos C) = 2 \sin B \sin C \cos a$.

Hence, on expanding the determinant, we find

$$\Sigma \sin^2 A \sin^2 x + 2\Sigma \sin B \sin C \cos a \sin y \sin z = -\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 4N^2.$$

Dividing across by N^2, and noting that

$$\frac{\sin A}{N} = \frac{\sin a}{n}, \text{ \&c.},$$

we obtain

$$\Sigma \sin^2 a \sin^2 x + 2\Sigma \sin b \sin c \cos a \sin y \sin z = 4n^2,$$

which reduces *in plano* to

$$ax + by + cz = 2\Delta.]$$

2. Find the angular radius of the circle cutting three small circles orthogonally in terms of their angular radii, r_1, r_2, r_3, and the angular distances, a, b, c, between their poles.

[Let α, β, γ be the angular distances of the pole of the orthogonal circle from the poles of the given circles. Then, if ρ be the unknown radius,

$$\cos\alpha = \cos\rho\cos r_1, \quad \cos\beta = \cos\rho\cos r_2, \quad \cos\gamma = \cos\rho\cos r_3.$$

Therefore

$$\begin{vmatrix} 1 & \cos c & \cos b & \cos\alpha \\ \cos c & 1 & \cos a & \cos\beta \\ \cos b & \cos a & 1 & \cos\gamma \\ \cos\alpha & \cos\beta & \cos\gamma & 1 \end{vmatrix} = 0 = \begin{vmatrix} 1 & \cos c & \cos b & \cos r_1 \\ \cos c & 1 & \cos a & \cos r_2 \\ \cos b & \cos a & 1 & \cos r_3 \\ \cos r_1 & \cos r_2 & \cos r_3 & \sec^2\rho \end{vmatrix},$$

which determines $\sec\rho$, as required].

3. Show that the arcs joining n points, $1, 2, 3, \ldots n$, on a sphere, are connected by the determinant relation

$$\begin{vmatrix} 1 & \cos\overline{12} & \cos 13 & \cdots & \cos\overline{1n} \\ \cos\overline{21} & 1 & \cos\overline{23} & \cdots & \cos\overline{2n} \\ \cos\overline{31} & \cos\overline{32} & 1 & \cdots & \cos\overline{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cos\overline{n1} & \cos\overline{n2} & \cos\overline{n3} & \cdots & 1 \end{vmatrix} = 0,$$

where $\overline{12}$, &c., denote the arcs joining the points 1, 2, &c.

[This relation is obtained in exactly the same manner as the relation among the connectors of four points was obtained in the foregoing Article.]

4. If O be any point on the sphere, and n_1, n_2, n_3 the functions of the sides of the triangles BOC, COA, AOB, corresponding to the function n of the triangle ABC, show that

$$n^2 = n_1^2 + n_2^2 + n_3^2 + 2n_2n_3\cos a + 2n_3n_1\cos b + 2n_1n_2\cos c.$$

[We have, $\sin a\sin x = 2n_1$, $\sin b\sin y = 2n_2$, $\sin c\sin z = 2n_3$; therefore, &c. by Ex. 1.]

133. **Expression for** $\cos\delta$ **with reference to any Triangle.**—In this Article we propose to find an expression for the angular distances δ, between two points P, P',

in terms of the angular distances of those points from the vertices of any given triangle ABC and the sides of that triangle.

Let the direction angles of A, B, C, the vertices of the given triangle, be $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2\ \gamma_2$; $\alpha_3, \beta_3, \gamma_3$; and ξ, η, ζ ; ξ', η', ζ', those of the two arbitrary points P, P', respectively. Then, by multiplication of determinants, we have

$$\begin{vmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 & 0 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 & 0 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 & 0 \\ \cos\xi' & \cos\eta' & \cos\zeta' & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 & 0 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 & 0 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 & 0 \\ \cos\xi & \cos\eta & \cos\zeta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos c & \cos b & \cos\alpha \\ \cos c & 1 & \cos a & \cos\beta \\ \cos b & \cos a & 1 & \cos\gamma \\ \cos\alpha' & \cos\beta' & \cos\gamma' & \cos\delta \end{vmatrix} = 0,$$

where α, β, γ ; α', β', γ' are the angular distances of the points in question from the vertices of the given triangle ABC. Expanding this determinant, we find

$$4n^2 \cos\delta = \begin{vmatrix} 1 & \cos c & \cos b & \cos\alpha \\ \cos c & 1 & \cos a & \cos\beta \\ \cos b & \cos a & 1 & \cos\gamma \\ \cos\alpha' & \cos\beta' & \cos\gamma & 0 \end{vmatrix}$$

$$= \Sigma \cos\alpha \cos\alpha' \sin^2 a + \Sigma (\cos\beta \cos\gamma' + \cos\beta' \cos\gamma) \sin b \sin c \cos A.\text{*}$$

---

[*] This expression for cos δ may also be obtained from the equation

$$d^2 = x^2 + y^2 + z^2 + 2yz \cos a + 2 \cos zx \cos b + 2\, xy \cos c,$$

which gives the diagonal d of a parallelopiped in terms of its edges, x, y, z, and their mutual inclinations, a, b, c. (See Chap. XV.)

This equation gives the angle between two lines in terms of the angles which they make with three fixed oblique axes (radii to A, B, C), and the angles a, b, c, between those axes. It obviously reduces to the determinant relation of Art. 132, when the two points coincide, for then $\delta = 0$, and $\cos\delta = 1$.

Examples.

1. Express the cosine of the angular distance between the poles of the in-circle and circumcircle of a triangle in terms of the sides of the triangle, and also in terms of the angular radii of the circles. (Cf. Art. 86.)

2. If $\theta_1, \theta_2, \theta_3 \ldots \theta_n$; $\theta_1', \theta_2', \ldots \theta_n'$ be the angular distances of two points, P, P', from n fixed points, 1, 2, 3, . . . n on the surface of a sphere, show that

$$\begin{vmatrix} 1 & \cos\overline{12} & \cos\overline{13} & \cdots & \cos\overline{1n} & \cos\theta_1 \\ \cos\overline{21} & 1 & \cos\overline{23} & \cdots & \cos\overline{2n} & \cos\theta_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \cos\overline{n1} & \cos\overline{n2} & \cos\overline{n3} & \cdots & 1 & \cos\theta_n \\ \cos\theta_1' & \cos\theta_2' & \cos\theta_3' & \cdots & \cos\theta_n' & 0 \end{vmatrix} = 0.$$

[Let $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; . . . $\alpha_n, \beta_n, \gamma_n$, be the direction angles of the fixed points; ξ, η, ζ; ξ', η', ζ', those of P and P', and proceed as in the foregoing Article, noticing that the coefficient of $\cos PP'$ vanishes by Art. 132, Ex. 3; and therefore zero may be substituted instead of $\cos PP'$, as above.]

134. **Problem.**—*If A_1, A_2, A_3, . . . A_n be any number of fixed points on the surface of a sphere, to find the locus of a point O, such that*

$$l_1 \cos OA_1 + l_2 \cos OA_2 + l_3 \cos OA_3 + \ldots + l_n \cos OA_n = \text{const.}$$

Let α, β, γ be the direction angles of O; α_1, β_1, γ_1, α_2, β_2, γ_2, &c., the direction angles of A_1, A_2, &c., with respect to a tri-quadrantal triangle ABC; we have

$$l_1 \cos OA_1 = l_1 (\cos\alpha \cos\alpha_1 + \cos\beta \cos\beta_1 + \cos\gamma \cos\gamma_1),$$

$$l_2 \cos OA_2 = l_2 (\cos\alpha \cos\alpha_2 + \cos\beta \cos\beta_2 + \cos\gamma \cos\gamma_2).$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$l_n \cos OA_n = l_n (\cos\alpha \cos\alpha_n + \cos\beta \cos\beta_n + \cos\gamma \cos\gamma_n).$$

Therefore, by addition,

$$\Sigma l_1 \cos OA_1 = \cos\alpha \Sigma l_1 \cos\alpha_1 + \cos\beta \Sigma l_1 \cos\beta_1 + \cos\gamma \Sigma l_1 \cos\gamma_1$$

$$= L \cos\alpha + M \cos\beta + N \cos\gamma = K, \qquad (1)$$

where L, M, N are known quantities, being $\Sigma l_1 \cos\alpha_1$ $\Sigma l_1 \cos\beta_1$, $\Sigma l_1 \cos\gamma_1$, respectively.

Now, if we find a point P, such that its direction cosines are

$$\frac{L}{G}, \ \frac{M}{G}, \ \frac{N}{G} \text{ (where } G^2 = L^2 + M^2 + N^2\text{)};$$

then, by Art. 129,

$$\cos OP = \frac{1}{G} (L \cos\alpha + M \cos\beta + N \cos\gamma).$$

Therefore, by (1),

$$\Sigma l_1 \cos OA_1 = G \cos OP = K.$$

This determines $\cos OP$, K and G being given quantities; and as P is a fixed point, the locus of O is a small circle round P as pole, the angular radius of the circle being given by the equation

$$\cos OP = \frac{K}{G}.$$

Cor. 1.—If $K = 0$, the locus of O is a great circle.

Cor. 2.—Supposing O to coincide with A, B, C successively, we have

$$\Sigma l_1 \cos AA_1 = L,$$
$$\Sigma l_1 \cos BA_1 = M,$$
$$\Sigma l_1 \cos CA_1 = N.$$

[This follows at once from (1)]

Cor. 3.—The point P is such that the sum of the cosines of its angular distances from A_1, A_2, A_3, &c., is a maximum.

[For in this case cos OP is unity, since O coincides with P, and therefore $\Sigma l_1 \cos PA_1 = G$. The diametrically opposite point gives a minimum value of sum of cosines.]

Hence, generally—*If the cosines of the angular distances of a point P from any number of fixed points are connected by a linear relation, the point is constrained to move on a circle.* And, in particular—*If the cosines of the angular distances of a point from the vertices of a triangle are connected by a linear relation, the locus of the point is a circle.* Or, what is the same thing—*If the sines of the angular distances of a point from the sides of a fixed triangle* (*or from any number of fixed arcs*) *are connected by a linear relation, the locus of the point is a circle;* the sides of the triangle (or fixed arcs) in this case being the polars of the fixed points in the preceding.

Example.

If a circle cut three given small circles at equal angles (α), the locus of its pole is a great circle.

Let δ_1, δ_2, δ_3 be the angular distances of the poles of the fixed circles from the pole of the variable circle; r_1, r_2, r_3 the angular radii of the former, and ρ that of the latter. Then

$$\cos\delta_1 = \cos\rho\cos r_1 + \sin\rho\sin r_1\cos\alpha;$$

also
$$\cos\delta_2 = \cos\rho\cos r_2 + \sin\rho\sin r_2\cos\alpha,$$

and
$$\cos\delta_3 = \cos\rho\cos r_3 + \sin\rho\sin r_3\cos\alpha.$$

Eliminating $\cos\rho$ and $\sin\rho\cos\alpha$, we obtain

$$\begin{vmatrix} \cos\delta_1 & \cos\delta_2 & \cos\delta_3 \\ \cos r_1 & \cos r_2 & \cos r_3 \\ \sin r_1 & \sin r_2 & \sin r_3 \end{vmatrix} = 0;$$

or $\cos\delta_1\sin(r_2 - r_3) + \cos\delta_2\sin(r_3 - r_1) + \cos\delta_3\sin(r_1 - r_2) = 0$, which is a linear relation connecting $\cos\delta_1$, $\cos\delta_2$, $\cos\delta_3$, in which the absolute term is zero, and therefore the pole describes a great circle (*Cor.* 1).

Miscellaneous Examples.

1. If $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$ be the direction angles of the vertices of a triangle *ABC*, prove that

$$\begin{vmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{vmatrix} = 2n.$$

[Square this determinant, and apply the relations

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = \cos\overline{12} = \cos c, \text{ \&c.},$$

and we obtain the determinant of footnote, Art. 30.]

2. If *ABC* be a tri-quadrantal triangle, prove that, using the notation of the previous Example,

$$\begin{vmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{vmatrix} = 1.$$

3. In the same case prove that

$$\begin{vmatrix} \cos\alpha_1 - 1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 - 1 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 - 1 \end{vmatrix} = 0.$$

[If Δ denote this determinant, and Δ' the determinant of Ex. 2, we have $\Delta = \Delta\Delta'$, which, by the multiplication of the determinants, reduces to $-\Delta$; therefore, &c.]

4. Apply the relation of Art. 132 to determine the radius of a sphere circumscribing a given tetrahedron in terms of its edges.

[The edge joining the points 1, 2 is equal to $2R\sin\frac{1}{2}(\overline{12})$. But $\cos\overline{12} = 1 - 2\sin^2\frac{1}{2}(\overline{12})$; therefore, &c.]

5. If A_1, A_2, A_3, &c., be fixed points, of which the direction angles are $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, &c., and if O be a point having direction angles α, β, γ, show that

$$\Sigma l_1 \cos^2 OA_1 = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma + 2F\cos\beta\cos\gamma + 2G\cos\gamma\cos\alpha + 2H\cos\alpha\cos\beta,$$

where

$$A = \Sigma l_1 \cos^2\alpha_1,\quad B = \Sigma l_1\cos^2\beta_1,\quad C = \Sigma l_1\cos^2\gamma_1,\quad F = \Sigma l_1\cos\beta_1\cos\gamma_1, \text{ \&c.}$$

[We have $l_1\cos^2 OA_1 = l_1(\cos\alpha\cos\alpha_1 + \cos\beta\cos\beta_1 + \cos\gamma\cos\gamma_1)^2$. Writing down the other terms, and adding, we find the above.]

6. In the same case, if O and O' be any two fixed points, show that

$$\Sigma l_1 \cos OA_1 \cos O'A_1 = A\lambda\lambda' + B\mu\mu' + C\nu\nu' + F(\mu\nu' + \mu'\nu) + G(\nu\lambda' + \nu'\lambda) + H(\lambda\mu' + \lambda'\mu),$$

where $\lambda\mu\nu$, $\lambda'\mu'\nu'$ are the direction cosines of O and O'.

7. By properly choosing the triangle of reference, the results of Examples 6 and 7 reduce respectively to

$$A\lambda^2 + B\mu^2 + C\nu^2,$$

and

$$A\lambda\lambda' + B\mu\mu + C\nu\nu'.$$

8. If a sphere be described round a regular polyhedron, prove that the sum of the cosines of the arcs joining any point on the sphere to the vertices of the polyhedron is zero.

9. A and B are fixed points on a sphere; P any point on the surface; show that a fixed point C can always be found in AB, or AB produced, such that

$$l \cos AP + m \cos BP = n \cos CP,$$

where l, m, n are constants. (Cf. Art. 40.)

10. If α, β, γ, δ be the arcs joining a point P to four fixed points on a sphere, find the locus of P when

$$\cos \alpha + \cos \beta + \cos \gamma + \cos \delta = \text{const.},$$

and determine when the locus becomes impossible.

(*London Univ. Exam. Papers.*)

11. If λ, μ, ν denote the perpendiculars from the vertices of any tri-quadrantal triangle on a transversal to the sides; prove the relation

$$\sin^2 \lambda + \sin^2 \mu + \sin^2 \nu = 1;$$

and hence show (see Art. 130) that

$$\sin^2 \delta = \Sigma (\cos \beta \cos \gamma' - \cos \beta' \cos \gamma)^2.$$

[Let PP' (fig. 48) be the transversal, and let it meet the sides of the triangle in the points X, Y, and Z. Then $\sin \lambda = \sin AZ \sin Z$ and $\sin \mu = \sin BZ \sin Z$. Squaring, and adding these equations,

$$\sin^2 \lambda + \sin^2 \mu = \sin^2 Z = \cos^2 \nu;$$

therefore $$\sin^2 \lambda + \sin^2 \mu + \sin^2 \nu = 1.$$

Again, applying the relation given in Misc. Ex. 24, of Chap. iii. to the quadrilateral $ABPP'$ (fig. 48),

$$\cos \alpha \cos \beta' - \cos \alpha' \cos \beta = \sin \delta \sin AB \cos Z = \sin \delta \sin \nu.$$

Hence

$$\Sigma (\cos \alpha \cos \beta' - \cos \alpha' \cos \beta)^2 = \sin^2 \delta (\sin^2 \lambda + \sin^2 \mu + \sin^2 \nu) = \sin^2 \delta.]$$

CHAPTER XI.

THEORY OF ANHARMONIC AND HARMONIC SECTION.

135. **Anharmonic Ratio.** — When four points, A, B, C, D, lie on a great circle they are connected by the relation

$$\sin BC \sin AD + \sin CA \sin BD + \sin AB \sin CD = 0.$$

Let the left member of this equation be divided in turn by each of its three terms, and it will be seen that there are thus six fractions formed, in pairs, the reciprocals of one another. Thus if λ, μ, ν denote one triad, the remaining triad may be denoted by $\frac{1}{\lambda}$, $\frac{1}{\mu}$, $\frac{1}{\nu}$.

Moreover, it will be seen, by writing at length the equations after each process of division, that λ, μ, ν are further connected by the equations

$$\frac{1}{\lambda} + \mu = \frac{1}{\mu} + \nu = \frac{1}{\nu} + \lambda = -1.$$

Therefore, if any one of the six ratios be given, all the others are completely determined.

Definitions.—Such a system of points is called a *Spherical Row;* and the system of arcs of great circles joining A, B, C, D to any point O on the sphere is a *Spherical Pencil,* or a *Pencil.* Each arc of the pencil is termed a *Ray.*

The functions λ, μ, ν, $\frac{1}{\lambda}$, $\frac{1}{\mu}$, $\frac{1}{\nu}$ are the six *Anharmonic Ratios* of the row of points A, B, C, D; but on account of one of them completely determining the remaining five, we shall in future deal with one as the *Anharmonic Ratio* of the row.

Let $\frac{\sin BC \sin AD}{\sin AB \sin CD}$ represent the anharmonic ratio of the row of points A, B, C, D, then similarly the function $\frac{\sin BOC \sin AOD}{\sin AOB \sin COD}$ may be taken to represent the anharmonic ratio of the pencil formed by joining O to the points.

The student will notice that the anharmonic ratio of a spherical row is equal to that of the plane pencil formed by joining the points to the centre of the sphere.*

We shall usually denote the pencil joining any point O to any other four points, A, B, C, D, by $O.ABCD$, and the row by $(ABCD)$.

136. **Harmonic Section.**—If the anharmonic ratio of a row (or pencil) be equal to unity, the row (or pencil) is said to be harmonic.

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* The six anharmonic ratios may be further expressed as the trigonometrical functions of an angle, thus:—Upon the segments AC and BD, as diameters, describe small circles, and join the poles M and N of these circles to either point of intersection O. Apply the relations of Art. 36, (5), (6), (7) to the angle MON, *e.g.* $\cos^2 \frac{1}{2} O = \frac{\sin \frac{1}{2} BC \sin \frac{1}{2} AD}{\sin \frac{1}{2} CA \sin \frac{1}{2} BD}$. Hence, for a triangle of sides double those of MON, and vertical angle θ, the anharmonic ratios will be found to be

$$\sin^2 \tfrac{1}{2}\theta, \quad \cos^2 \tfrac{1}{2}\theta, \quad -\tan^2 \tfrac{1}{2}\theta, \quad \operatorname{cosec}^2 \tfrac{1}{2}\theta, \quad \sec^2 \tfrac{1}{2}\theta, \quad -\cot^2 \tfrac{1}{2}\theta.$$

(See Cremona's *Projective Geometry*, Art. 72 (V.);
Casey's *Conics*, p. 66, Ex. 32.)

In this case we obviously have

$$\frac{\sin AB}{\sin BC} = \frac{\sin AD}{\sin DC};$$

that is, the ratio of the sines of the segments into which B divides AC is the same as the ratio of the sines of the segments into which D divides it. B dividing AC internally, and D dividing it externally, we shall say that *when two points, B and D, divide an arc AC harmonically, they cut the arc internally and externally in the same ratio of sines.*

137. **Theorem.**—*The anharmonic ratio of a spherical pencil is the same as the anharmonic ratio of the four points in which any transversal great circle meets it.*

Let a great circle meet a pencil drawn from O (fig. 49) in points A, B, C, D; then if p be the perpendicular from O on AD, we have, from the triangle AOB,

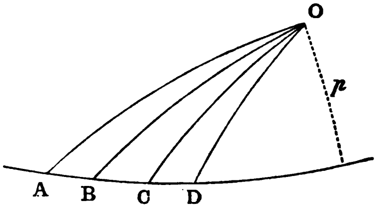

Fig. 49.

$$\sin OA \sin OB \sin AOB = \sin AB \sin p. \quad (1)$$

Similarly, $\sin OC \sin OD \sin COD = \sin CD \sin p. \quad (2)$

Also $\sin OB \sin OC \sin BOC = \sin BC \sin p; \quad (3)$

and $\sin OA \sin OD \sin AOD = \sin AD \sin p. \quad (4)$

Dividing the product of (3) and (4) by the product of (1) and (2), we find

$$\frac{\sin BOC \sin AOD}{\sin AOB \sin COD} = \frac{\sin BC \sin AD}{\sin AB \sin CD}$$

Hence the anharmonic ratio of the four points in which any transversal great circle meets a fixed pencil of four arcs is constant, and equal to the anharmonic ratio of the pencil.

Cor. 1.—A harmonic pencil cuts any great circle in a harmonic row.

Cor 2.—If two pencils intersect in four points on a great circle, their anharmonic ratios are equal.

Cor. 3.—If the angles contained by corresponding rays of two pencils be equal, the anharmonic ratios of the pencils are equal.

Remark.—By the application of the above method we may establish the following more general Theorem:—*If in a figure on a sphere we have a function consisting of the product of the sines of one set of arcs divided by the product of the sines of another set of arcs, each arc of one set being part of the same great circle with one of the other set, and the whole system of extremities of arcs being the same for both sets; then the function will remain unaltered in value, if in place of the arcs we substitute the angles they subtend at any point on the sphere.*

Cor. 4.—If from any point on the sphere perpendiculars be drawn to the rays of a given pencil, they form a pencil equianharmonic with the given one.

[For the perpendiculars pass through the poles of the rays of the given pencil, but the poles are concyclic, and form a row equianharmonic with the given pencil.]

138. *When three points of a spherical row are given, along with its anharmonic ratio, the fourth point is determined if its relative position be also known* (*that is between which successive pair of points it lies*). For the problem is

reducible to the following:—Given the sum or difference of two angles, and the ratio of their sines; find the angles. Since the ratio of sines is known, we know the ratio tan (half sum) : tan (half difference), which, with the given sum or difference, determines the angles.

In the same manner it follows, that if *three rays of a spherical pencil be given, the fourth is determined if the anharmonic ratio of the pencil and the relative position of the fourth ray be known.*

Cor. 1.—Three rays of a harmonic pencil (or points of a harmonic row) determine it (the relative position of the fourth being known).

Cor. 2.—If two pencils have a common vertex and equal anharmonic ratios, and be such that three rays of one are the production of three rays of the other; then the fourth ray of the former will be the production of the fourth ray of the latter (or coincide with it).

139.—**Theorem.**—*If two equianharmonic pencils have a common ray, the intersections of the remaining three corresponding pairs are concyclic.*

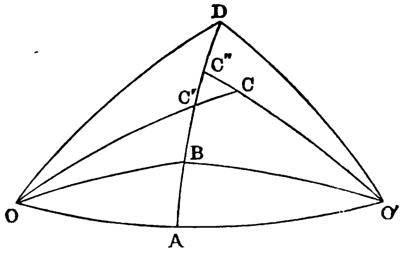

Fig. 50.

Let OO' (fig. 50) be the common ray, and suppose B, C, D to be the intersections of the corresponding rays. Let

BD meet OC and $O'C$ in C' and C'', respectively. Then, by hypothesis,

$$(ABC'D) = (ABC''D).$$

But by the foregoing Article this is impossible, unless C' coincides with C''. Therefore DB passes through C.

Cor.—If A, B, C; A', B', C', be two triads of points on two great circles intersecting at O, and if the anharmonic ratio of the rows $(OABC)$, and $(OA'B'C')$ be equal, the arcs AA', BB', CC' will be concurrent. [For, join O to the intersection of AA' and BB'; then the arc from this point to C must pass through C' if the condition of the problem holds.]

We apply this result to the following theorem.

140. **Triangles in Perspective, or Copolar Triangles.**—*If two spherical triangles be such that the arcs joining corresponding vertices meet in a point,[*] the intersections of corresponding sides will lie on a great circle.*[†]

Let ABC and $A'B'C'$ (fig. 51) be the triangles, O the

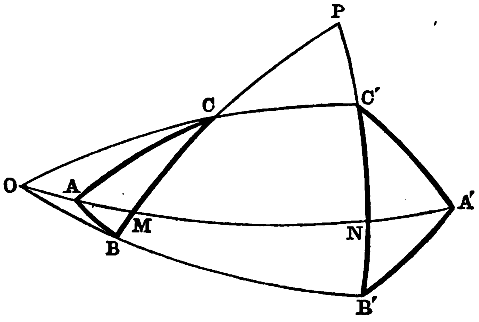

Fig. 51.

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[*] Called the centre of perspective, or pole of the triangles.

[†] Called the axis of perspective, or axis of homology of the triangles.

point of intersection of arcs joining corresponding vertices, P the intersection of BC and $B'C'$.

Then $$O \,.\, PCMB = A \,.\, PCMB,$$

and $$O \,.\, PCMB = O \,.\, PC'NB' = A' \,.\, PC'NB'.$$

Therefore $$A \,.\, PCA'B = A' \,.\, PC'AB'.$$

The ray AA' is common to both these pencils, and therefore the intersections of their corresponding rays, AP, $A'P$; AC, $A'C'$; AB, $A'B'$, lie on a great circle; therefore, &c.

Cor.—If two triangles be such that the intersections of corresponding sides are concyclic; then the arcs joining corresponding vertices will be concurrent.

141. The student may observe that the proof of the proposition of the foregoing Article is identical with that given for plane triangles in perspective, and this remark may be applied to much that has gone before. Hence the following propositions will present no difficulty once the corresponding theorems *in plano* have been mastered (see Casey's *Sequel*, Book VI.).

(1) *When three triangles are two by two in perspective, and have the same axis of perspective, their three centres of perspective are concyclic.*

(2) *When three triangles are two by two in perspective, and have the same centre of perspective, their three axes of perspective are concurrent.*

142. **Harmonic Properties of a Spherical Triangle.**—In the fig. of Art. 125, we have three arcs, AA', BB', CC', drawn through the vertices of a spherical triangle,

and intersecting in a point O, and we found there that the arcs $A'B'$, $B'C'$, $C'A'$ meet the opposite sides in points A'', B'', C'', which divide them externally in the same ratio of sines, as A', B', C' divide them internally. Hence the rows

$$(BCA'A''),\ (CAB'B''),\ (ABC'C'')$$

are each harmonic, and therefore also the pencils

$$A\,.\,BCA'A'',\ B\,.\,CAB'B'',\ C\,.\,ABC'C'',$$

are each harmonic.

143. **Harmonic Properties of a Complete Quadrilateral.**—The same figure is easily seen to be a complete quadrilateral, of which AA', BB', CC'' are the three diagonals.

Hence, since $C\,.\,AC'BC''$ is harmonic, it follows that a pair of opposite sides of a complete quadrilateral form a harmonic pencil with the third diagonal, and the arc joining their intersection to the intersection of the other two diagonals. Also—*The two diagonals of a complete quadrilateral form a harmonic pencil with the arcs joining their intersection to the extremities of the third.*

144. *If $ABCD$ be a harmonic row,*

$$\tan AB,\ \tan AC,\ \tan AD,$$

or the tangents of the arcs measured from A, are in harmonical progression.

For

$$\sin BC \sin AD = \sin AB \sin CD;$$

or

$$\sin (AC - AB) \sin AD = \sin AB \sin (AD - AC).$$

Dividing across by

$$\sin AB \sin AC \sin AD,$$

we get

$$\cot AB - \cot AC = \cot AC - \cot AD.$$

Therefore

$$\cot AB + \cot AD = 2 \cot AC;$$

that is, the cotangents of the arcs are in arithmetical progression, and hence their tangents are in harmonic.

145. *If $(ABCD)$ be a harmonic row, and if either of the mean arcs AC or BD be bisected; the tangents of the arcs, measured from the point of bisection to the other points of section of the great circle, are in geometric progression.*

For, let M be the middle point of AC. Then

$$\frac{\sin AB}{\sin BC} = \frac{\sin AD}{\sin CD}.$$

Therefore

$$\frac{\sin AB - \sin BC}{\sin AB + \sin BC} = \frac{\sin AD - \sin CD}{\sin AD + \sin CD};$$

or,

$$\frac{\tan \frac{1}{2}(AB - BC)}{\tan \frac{1}{2}(AB + BC)} = \frac{\tan \frac{1}{2}(AD - CD)}{\tan \frac{1}{2}(AD + CD)};$$

or,

$$\frac{\tan BM}{\tan CM} = \frac{\tan CM}{\tan DM}.$$

Therefore

$$\tan BM \tan DM = \tan^2 CM.$$

Otherwise thus—Let $(ABCD)$ be a harmonic row on a great circle. Draw a tangent to it at A, and produce the radii to B, C, D, to meet the tangent in B', C', D'. Then the pencil $O \,.\, ABCD$ is a harmonic pencil, and therefore

$(AB'C'D')$ is a harmonic row, O being the centre of the sphere. If we take the radius of the sphere as unity, we will have AB', AC', AD' equal to $\tan AB$, $\tan AC$, $\tan AD$, respectively; but since $AB'C'D'$ is a harmonic row, AB', AC', AD' are in harmonic progression; therefore $\tan AB$, $\tan AC$, $\tan AD$ are in harmonic progression.

Again, if M be the middle point of AC, and if a tangent be drawn at it to meet the pencil $O\,.\,ABCD$ in $A''B''C''D''$; then this latter row will be harmonic, and M will be the middle point of $A''C''$. Therefore

$$B''M\,.\,D''M = (C''M)^2;$$

or

$$\tan BM\,.\,\tan DM = \tan^2 CM.$$

Examples.

1. The internal and external bisectors of an angle of a triangle form a harmonic pencil with the sides containing the angle; and, conversely, if a pair of rays of a harmonic pencil are at right angles, they will bisect internally and externally the angle between the other two rays.

2. Hence show that the arc joining the middle points of any pair of sides of a triangle meets the third side at 90° from its middle point.

[The radii to the extremities of any side, its middle point, and the point where the arc in question meets it, form a plane harmonic pencil, and one pair bisects internally and externally the angle between the other pair.]

NOTE.—By referring to fig. 46, we see that if A' and C' be the middle points of the sides, $A'A''$ and $C'C''$ will be quadrants; but in this case B' will be the middle point of AC; and therefore $B'B''$ will also be a quadrant. Hence, by considering the quadrilateral $C'A'C''A''$, we find that *if two diagonals, $A'A''$ and $C'C''$, of a quadrilateral be quadrants, the third diagonal, $B'B''$ will also be a quadrant.*

3. Given in magnitude and position the bases AB and $A'B'$ of two triangles; if the vertex of each moves on the base of the other, the arc joining the intersections of corresponding sides passes through a fixed point.

[Let AB and $A'B'$ be the fixed bases, O, the intersection of AB' and $A'B$, is a fixed point. Then we have

$$O\,.\,CA'MA = A'\,.\,PBC'A,$$
$$O\,.\,CBNB' = B'\,.\,PBC'A.$$

Therefore $$O . CA'MA = O . CBNB';$$

and as three of the rays of one of these pencils are the productions of three of the rays of the other, the fourth rays, OM and ON, must be parts of the same great circle. Therefore MON is a great circle. That is, MN passes through a fixed point O.

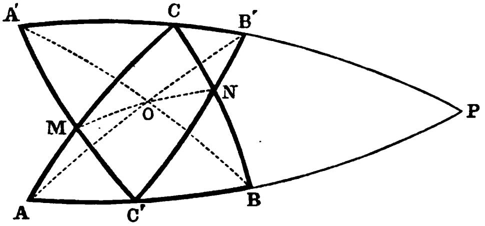

Fig. 52.

Similarly, the arc joining the intersections of non-corresponding sides passes through a fixed point, viz., the intersection of AA' and BB'.]

[Note.—This is a particular case of Pascal's Theorem.]

4. Hence show that if a quadrilateral be divided into two others by an arc of a great circle drawn across it, the arc joining the intersection of the diagonals of one of these quadrilaterals to the intersection of the diagonals of the other passes through the intersection of the diagonals of the original quadrilateral.

5. If the vertices of a variable triangle move on three fixed great circles, and if two of the sides pass through two fixed points, concyclic with the intersection of the great circles on which the opposite vertices move, show that the third side also passes through a fixed point.*

[In the fig. of Ex. 3 consider all the arcs fixed except $A'C'$, $B'C$, OC', and MN. Then the triangle MNC' obviously satisfies the problem.]

6. If the sides of a variable triangle pass through three fixed points, and if two of the vertices move on two great circles, the intersection of which lies on the great circle joining the points through which the opposite sides pass, the locus of the third vertex is a great circle.*

(See fig. of Ex. 3.)

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* One of two.

7. If the sides of a variable triangle pass through three fixed concyclic points, and if two of the vertices move on two given great circles, the locus of the third vertex will be a great circle* passing through the intersection of the given circles.

Take any three positions of the triangle, then it is easily seen that we have three pencils, one from each of the fixed points, having a common ray, and equal anharmonic ratios. These pencils intersect on the two given circles and on the locus of the third vertex, which, by Art. 139, is a great circle, as in question.]

8. If the three vertices of a triangle move on three concurrent great circles, and if two of the sides pass through two given points, the third side will pass through* a fixed point concyclic with the given ones.

9. Given in position a pair of opposite sides of a quadrilateral, and the point of intersection of the other pair; find the locus of the intersection of the diagonals.

Ans. A great circle—the harmonic conjugate of the third diagonal with respect to the pair of given sides.

10. If from the extremities of the base of a spherical triangle arcs be drawn so as to intersect on the perpendicular to the base from the opposite vertex, the arcs joining the foot of the perpendicular to their intersections with the opposite sides are equally inclined to the sides.

[Referring to fig. 46, Art. 125, the pencil $C'.B''CB'A$ is harmonic; and if the pair of rays $C'C$ and $C'A$ are at right angles, the rays $C'B'$ and $C'B''$ will bisect the angle between them internally and externally.]

Hence the pedal triangle of a spherical triangle is such that its sides are equally inclined to the sides of the original triangle;† *and therefore, as in plano, its perimeter is less than the perimeter of any other triangle which can be inscribed in the given one.*

11. If a great circle revolve round a fixed point O, and cut two given great circles in points A and B, find the locus of a point P on the revolving circle such that

$$\cot OA + \cot OB = 2\cot OP.$$

Ans. A great circle passing through their intersection and the harmonic conjugate, with respect to them, of the arc joining O to their intersection.

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* One of two.

† See also fig. 34, where ABC is the pedal triangle of $O_1O_2O_3$.

12. If a great circle revolve round a fixed point O, and cut any number of fixed great circles in points A_1, A_2, A_3, . . . find the locus of a point P on the revolving circle such that

$$\cot OP = \cot OA_1 + \cot OA_2 + \cot OA_3 + \ldots, \text{\&c.}$$

[By successive applications of Ex. 11, the locus is easily seen to be a great circle.]

13. Describe a triangle having its vertices on three given great circles, and its sides passing through three given points.

(See Exs. 7 and 8.)

14. In a given spherical polygon inscribe another of the same number of sides, so that each side may pass through a given point.

SPHERICAL INVOLUTION.

146. Those who have read plane geometry will be aware that if from a fixed point O, on a right line, lengths OA, OA'; OB, OB', &c., be measured off in pairs such that

$$OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC' = \text{\&c.} = \kappa^2;$$

when κ is some constant quantity, then the system of points will be so related that the anharmonic ratio of any four of them will be equal to the anharmonic ratio of their four conjugates (A being conjugate to A', B to B', &c.). Such a system of points is said to be in involution, and from this point of view spherical involution will present no difficulty.

Definition.—If three pairs of points, A, A'; B, B'; C, C', on a great circle, be such that the anharmonic ratio of any four (not being two pairs of conjugate points) is the same as that of their four conjugates (the relative order of the two sets being the same), then every set of four will possess a similar property. Three such pairs of points are said to be in *Spherical Involution.*

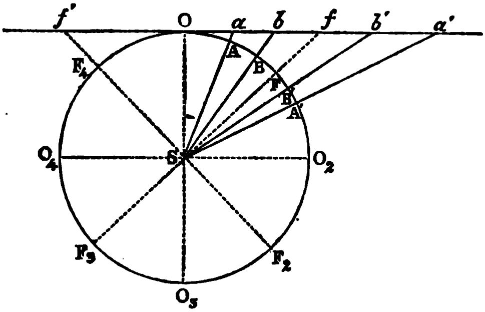

Fig. 53.

147. If a, a'; b, b', &c. (fig. 53), be a system of points in involution on a right line, and if Of be taken equal to κ (where $\kappa^2 = Oa \, . \, Oa'$), the point f, being its own conjugate, is called a *Focus*, or *Double Point*, and O is called the *Centre* of the involution. Now if OAB be a great circle on a sphere, the plane of the circle coinciding with the plane of the paper, Oa a tangent to it at O, the pencil joining S, the centre of sphere, to a, a'; b, b', &c., is in involution, since the anharmonic ratio of any four of its rays is the same as that of their four conjugates. Again, the anharmonic ratio of any four of the rays of this pencil is the same as that of the four points in which they intersect the great circle.* Therefore the points A, A'; B, B', &c., in which this pencil meets the great circle, are such

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* If A, B, C, D be these points of section, both ratios are equal to $\frac{\sin BC \sin AD}{\sin AB \sin CD}$.

that the anharmonic ratio of any four of them is equal to that of their four conjugates; that is, they form a system in involution.

The ray Sf to the focus f is called a *Double Ray*, or *Focal Ray*. It obviously meets the great circle in a point F, which is its own conjugate, which is therefore called a *Focus*, or *Double Point*, of the spherical involution. So, also, as *in plano*, the point O is termed the *Centre* of the spherical involution.

To the system of points a, b, &c., corresponds a similar system on the opposite side of O, having a focus f' such that $Of' = Of = \kappa$. Similarly, on the sphere to the system A, B, C, &c., will correspond a similar system, which are the reflections of them with respect to the diameter OS, having a focus F_4 such that $OF_4 = OF$. Thus on the sphere there are *four foci*, viz., F, F_4, and their diametrically opposite points, F_3, F_2, and *four centres* of involution, viz., O, O_2, O_3, O_4, the points bisecting the angular distances between the foci.

The pencil of great circles joining any point on the sphere to a system of points in involution forms a *pencil in involution*, since obviously it is such that the anharmonic ratio of any four of its rays is equal to that of their four conjugates (Art. 137).

If the radius of the sphere be unity, we have

$$Oa = \tan OA, \quad Ob = \tan OB, \text{ \&c.};$$

and hence, in a spherical involution,

$$\tan OA \tan OA' = \tan OB \tan OB' = \text{\&c.} = \tan^2 OF.$$

Examples.

1. The point conjugate to the centre O of an involution is 90° distant from it.

2. Any great circle drawn across a pencil in involution is cut in a system of points in involution. (Art. 137.)

3. Any pair of conjugate points such as A, A' are harmonic conjugates with respect to any pair of foci not diametrically opposite.

4. Three pairs of points having a common pair of harmonic conjugates form a system in involution.

[Ex. 3 and 4 follow at once from the corresponding properties on the line.]

5. The six arcs drawn from any point on the surface of the sphere to the intersections of the sides of a spherical quadrilateral form a pencil in involution.

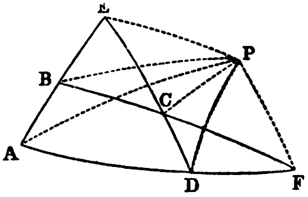

Fig. 54.

[Let P (fig. 54) be the point, $ABCDEF$ the quadrilateral. Then

$$P\,.\,BACF = A\,.\,BPCF = A\,.\,EPCD.$$

Hence $$P\,.\,BACF = P\,.\,DCAE;$$ therefore, &c.]

6. Any great circle drawn across the sides and diagonals of a spherical quadrilateral is cut in involution.

CHAPTER XII.

THE SMALL CIRCLE.

SECTION I.

Introductory.

148. **Theorem.**—*If a spherical polygon of any even number of sides be inscribed on a small circle, and if from any point O on the circle perpendiculars* $p_1, p_2, \ldots p_{2n}$ *be drawn to the sides* $a_1, a_2, \ldots a_{2n}$, *respectively, then*

$$\frac{\sin p_2 \sin p_4 \ldots \sin p_{2n}}{\sin p_1 \sin p_3 \ldots \sin p_{2n-1}} = \frac{\cos \frac{1}{2} a_1 \cos \frac{1}{2} a_3 \ldots \cos \frac{1}{2} a_{2n-1}}{\cos \frac{1}{2} a_2 \cos \frac{1}{2} a_4 \ldots \cos \frac{1}{2} a_{2n}};$$

that is, the continued product of the sines of the even perpendiculars bears a constant ratio to the continued product of sines of the odd perpendiculars, as O moves round the circle. Denote the angles which the sides a_1, a_2, &c., subtend at O by O_1, O_2, &c. The triangle BOC gives

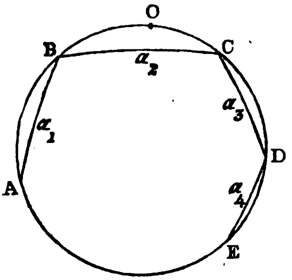

Fig. 55.

$$\sin OB \sin OC \sin O_2 = \sin a_2 \sin p_2. \qquad (1)$$

But if R be the circumradius of any triangle, we have, by Art. 83 (4),

$$2 \tan R \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c = \sin a \div \sin A.$$

Therefore (1) becomes

$$\sin OB \sin OC = 2 \tan R \cos \tfrac{1}{2} OB \cos \tfrac{1}{2} OC \cos \tfrac{1}{2} a_2 \sin p_2;$$

or, $$2 \sin \tfrac{1}{2} OB \sin \tfrac{1}{2} OC = \tan R \cos \tfrac{1}{2} a_2 \sin p_2.$$

Similarly, $$2 \sin \tfrac{1}{2} OD \sin \tfrac{1}{2} OE = \tan R \cos \tfrac{1}{2} a_4 \sin p_4.$$

. . . .

&c. = &c.

. . . .

Writing down the corresponding equations for the odd sides and perpendiculars, and noting that the continued product of the left-hand members is the same for each set, we find

$$\frac{\sin p_2 \sin p_4 \ldots \sin p_{2n}}{\sin p_1 \sin p_3 \ldots \sin p_{2n-1}} = \frac{\cos \frac{1}{2} a_1 \cos \frac{1}{2} a_3 \ldots \cos \frac{1}{2} a_{2n-1}}{\cos \frac{1}{2} a_2 \cos \frac{1}{2} a_4 \ldots \cos \frac{1}{2} a_{2n}}.$$

Cor. 1.—If the polygon be regular, the product of the sines of one set of perpendiculars is equal to the product of the sines of the other.

Cor. 2.—If a spherical quadrilateral be inscribed in a circle,

$$\frac{\sin p_2 \sin p_4}{\sin p_1 \sin p_3} = \frac{\cos \frac{1}{2} a_1 \cos \frac{1}{2} a_3}{\cos \frac{1}{2} a_2 \cos \frac{1}{2} a_4}.$$

Cor. 3.—If we suppose two opposite sides, a_2, a_4, of the quadrilateral to coincide, the other sides, a_1, a_3, become tangents to the circle, the coincident sides being their chord of contact, c, and

$$\frac{\sin p_1 \sin p_3}{\sin^2 p_2} = \cos^2 \tfrac{1}{2} c.$$

Hence, if two tangent great circles be drawn to a small circle, the ratio of the product of the sines of the perpendiculars from any point of the circle on the tangents, to the square of the sine of the perpendicular on their chord of contact, is constant, and equal to the square of the cosine of half the chord of contact.

Cor. 4.—By supposing two consecutive vertices of the polygon to coincide, we obtain a theorem for a polygon of an odd number of sides.

Cor. 5.—If we suppose a hexagon reduced to a triangle by the coincidence of its vertices in pairs, we have

$$\frac{\text{product of sines of perps. on tangents at vertices}}{\text{product of sines of perps. on sides}} = \cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c;$$

and for a polygon of n sides this ratio is equal to

$$\cos\tfrac{1}{2}a_1\cos\tfrac{1}{2}a_2\cos\tfrac{1}{2}a_3\ \ldots\ \cos\tfrac{1}{2}a_n.$$

Cor. 6.—*In plano*, we have

$$p_1p_3p_5\ \ldots\ p_{2n-1} = p_2p_4\ \ldots\ p_{2n}.$$

149. All the results of the foregoing Article may be deduced from *Cor.* 3; and, as it is of considerable importance, we append the following simple proof of it:—Let tangents be drawn at the vertices, A, B, of a triangle to the circumcircle, and let the perpendiculars from C on the tangents, and their chord of contact, c, be p_1, p_2, p, respectively. Then, if H be the pole of the circumcircle, we have

$$\frac{\sin p_2}{\sin p} = \frac{\cos CBH}{\sin B} = \frac{\cos(S-A)}{\sin B}.$$

Similarly,

$$\frac{\sin p_1}{\sin p} = \frac{\cos (S-B)}{\sin A}.$$

Therefore

$$\frac{\sin p_1 \sin p_2}{\sin^2 p} = \frac{\cos (S-A) \cos (S-B)}{\sin A \,.\, \sin B} = \cos^2 \tfrac{1}{2} c.$$

Remark.—By drawing tangents at the vertices of a quadrilateral inscribed in a small circle *Cor.* 2 easily follows; and similarly we may obtain the other results in Art. 148.

Definition.—The anharmonic ratio of the pencil joining any variable point on a small circle to four fixed points on it is called the Anharmonic Ratio of the four points.

150. **Theorem.**—*The anharmonic ratio of four fixed points on a small circle is constant.*

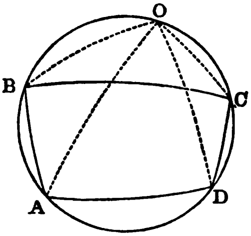

Fig. 56.

Let A, B, C, D be the fixed points; O any variable point on the circle. Then the triangles BOC and AOD give us

$$\sin OB \sin OC \sin BOC = \sin BC \sin p_2,$$

and

$$\sin OA \sin OD \sin AOD = \sin AD \sin p_4,$$

where p_2 and p_4 are the perpendiculars from O on the sides BC and AD. If p_1 and p_3 be the perpendiculars on the remaining side, we will have, by writing down, for the triangle AOB and DOC, the expressions corresponding to the above,

$$\frac{\sin BOC \sin AOD}{\sin AOB \sin COD} = \frac{\sin BC \sin AD \sin p_2 \sin p_4}{\sin AB \sin CD \sin p_1 \sin p_3}. \tag{1}$$

But by Art. 148,

$$\frac{\sin p_2 \sin p_4}{\sin p_1 \sin p_3} = \frac{\cos \frac{1}{2} AB \cos \frac{1}{2} CD}{\cos \frac{1}{2} BC \cos \frac{1}{2} AD}.$$

Therefore, by substituting in (1) and reducing,

$$\frac{\sin BOC \sin AOD}{\sin AOB \sin COD} = \frac{\sin \frac{1}{2} BC \sin \frac{1}{2} AD}{\sin \frac{1}{2} AB \sin \frac{1}{2} CD};$$

that is, *the anharmonic ratio of four fixed points on a small circle is constant.*

Cor.—In the same manner we have, generally, using the notation of Art. 148,

$$\frac{\sin O_2 \sin O_4 \ldots \sin O_{2n}}{\sin O_1 \sin O_3 \ldots \sin O_{2n-1}} = \frac{\sin \frac{1}{2} a_2 \sin \frac{1}{2} a_4 \ldots \sin \frac{1}{2} a_{2n}}{\sin \frac{1}{2} a_1 \sin \frac{1}{2} a_3 \ldots \sin \frac{1}{2} a_{2n-1}}.$$

Examples.

1. *Pascal's Theorem on the Sphere.* If a spherical hexagon be inscribed in a small circle, the intersections of the opposite sides, viz., 1 and 4, 2 and 5, 3 and 6, are concyclic.

[Proof, as *in plano.*]

2. Show, from Ex. 1, that if a pentagon be inscribed in a small circle, and a tangent be drawn at one angle to meet the opposite side, and if the sides containing that angle be produced to meet the remaining pair of sides, the three points of section so formed lie on a great circle.

3. If a quadrilateral be inscribed in a small circle, tangents at the extremities of either diagonal intersect on the third diagonal.

[A particular case of Ex. 1.]

4. The tangents to a small circle at the vertices of an inscribed triangle meet the opposite sides in three points, which are concyclic.

5. The sixty great circles corresponding to the sixty Pascal's lines, *in plano*, which result from taking six points in any order, as the vertices of a hexagon inscribed in a circle, consist of twenty concurrent triads.

6. If two triangles, ABC, $A'B'C'$, be inscribed in a small circle, any two sides, AB and $A'B'$, are divided equianharmonically by the remaining four.

[For $C.A'BAB' = C'.A'BAB'$; therefore, &c.]

7. If three great circles be drawn from a point cutting a given small circle, show that the pencil joining any point on the small circle to the six points of section is in involution.

8. A great circle drawn to meet a small circle, and the sides of a quadrilateral inscribed in it, is cut in involution.

(*a*) The points where the transversal meets the diagonals belong to the involution.

9. If the angular points of a quadrilateral inscribed in a small circle form a harmonic system of points, prove that the product of the sines of the halves of the diagonals is equal to twice the product of the sines of the halves of a pair of opposite sides.

10. If δ be the angular distance of a fixed point, O, from the pole of a small circle of spherical radius, α, prove that the angular distance, ρ, of any variable point on the circle from O, is connected with the perpendicular, p, from O, on the tangent to the circle at the variable point, by the equation

$$\cos\rho + \sin p \tan\alpha = \cos t,$$

where t is the tangent from the fixed point to the circle.

[$\cos\delta = \cos\rho\cos\alpha + \sin\rho\sin\alpha\cos\widehat{\rho\alpha}$. But $\sin p = \sin\rho\cos\widehat{\rho\alpha}$;

therefore $\cos\delta = \cos\rho\cos\alpha + \sin\alpha\sin p$, &c.]

SECTION II.

Theory of Poles and Polars.

151. **Definitions.**—If on any spherical radius of a small circle two points be taken, such that the product of the tangents of their angular distances from the pole of the circle is equal to the square of the tangent of its angular radius, then a great circle, perpendicular to the radius through either point, is called the *Polar* of the other, with respect to the circle, and each point is termed the *Pole* of the perpendicular through the other.*

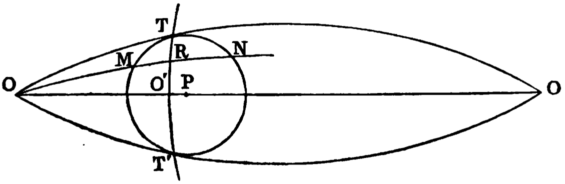

Fig. 57.

Thus if P (fig. 57) be the pole of a small circle of spherical radius, r, and if O and O' be two points on a great circle, through P, such that

$$\tan PO \tan PO' = \tan^2 r, \qquad (1)$$

then the great circle TT', through O', perpendicular to OP, is the polar of O, and O is the pole of TT', with respect to the small circle; similarly, O', and a great circle through O, perpendicular to OP, are pole and polar. The great circle OP is termed the *Polar Axis* of O, and the foot of the perpendicular from P, on any great circle, is called the *Polar Centre* of that circle, with respect to any small circle described with P as pole.

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* The terms "pole" and "polar" thus applied need not give rise to any ambiguity of expression, notwithstanding that they have been defined in a different sense in Art. 4. In that Article the point and circle are pole and polar, *with respect to one another;* here they are pole and polar, *with respect to the small circle.*

Two points, so situated that each is on the polar of the other, are called *Conjugate Points.* Thus the polar of a point is the locus of the entire system of its conjugates, and the polar centre of any chord of a circle is at its middle point.

In general, if three points be situated on a great circle in any manner, with respect to one another, any one of them may be defined as the pole of the perpendicular to the great circle through either of the others, with respect to a small circle (real or imaginary) described round the third as pole. The radius of the circle being given by equation (1) will be real or imaginary, according as $\tan PO \tan PO'$ is positive or negative; and this in turn depends on the magnitudes of PO and PO', and also on the relative positions of O and O', with respect to P—two arcs being affected with opposite signs if measured in opposite directions.

Before entering into the two cases thus presented, we shall first show that the polar of a point, with respect to a circle, passes through the points of contact of the tangents from the point to the circle. Join T to O and P (fig. 57). Then since $\tan PO \tan PO' = \tan^2 PT$, it follows that the angle PTO is right; and therefore OT touches the circle at T. *Hence the polar of a point, with respect to a small circle, may be regarded as the spherical chord of con-*

tact of the tangents from the point to the circle. From this point of view the student may easily investigate the relative positions of O and O', with respect to P, as PO and PO' vary in magnitude, the circle remaining *real.*

The following Examples will be found to contain the most important cases:—

Examples.

1. If O and O' are on the same side of P, the circle will be *real* when PO and PO' are each less than, or each greater than, a quadrant; and *imaginary* when either is less than a quadrant, the other being greater.

2. If O and O' are on opposite sides of P, the circle will be *real* when either PO or PO' is less than a quadrant, the other being greater; and *imaginary* when they are each less than, or each greater than, a quadrant.

3. In Ex. 1, the points O and O' being on the same side of P and PO and PO', each greater than a quadrant, O and O' are both outside the circle, and are *diametrically opposite* to the case in which PO and PO' are each less than a quadrant, in which case one is inside, and the other outside, the circle.

4. When O is on the circle O' it coincides with it, and the polar becomes the tangent at the point.

5. When O coincides with P, PO' is a quadrant; and thus *the polar of P with respect to any small circle described round it, is the great circle, of which P is the pole.*

6. If one point is inside the circle the other is outside it.

7. The points in which PO meets the circle are harmonic conjugates, with respect to O and O'.

[For $\tan PO \tan PO' = \tan^2 r$.]

8. If two small circles intersect each other orthogonally, their common chord is the polar of the spherical centre of either, with respect to the other, and conversely.

9. If A and B be two points (fig. 58), and if from A a perpendicular, AM, be drawn to the polar of B, and from B a perpendicular, BN, to the polar of A, prove that

$$\frac{\sin AM}{\sin BN} = \frac{\sin PB' \cos PA}{\sin PA' \cos PB},$$

where P is the pole of the small circle, and A' and B' the points where the polars of A and B meet PA and PB.

[Let AM and PB meet in Q, then $QM = QB' = 90°$; and if BN and PA' meet in R, $RN = RA' = 90°$. Therefore

$$\frac{\sin AM}{\sin BN} = \frac{\cos AQ}{\cos BR} = \frac{\cos PA \cos PQ\,(1 + \tan PA \tan PQ \cos P)}{\cos PB \cos PR\,(1 + \tan PB \tan PR \cos P)}.$$

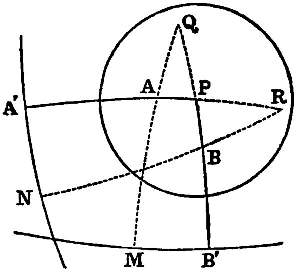

Fig. 58.

But

$\tan PA \tan PQ = \tan PB \tan PR$, since $\tan PA \tan PA' = \tan PB \tan PB'$.

Therefore

$$\frac{\sin AM}{\sin BN} = \frac{\cos PQ \cos PA}{\cos PR \cos PB} = \frac{\sin PB' \cos PA}{\sin PA' \cos PB}.\Big]$$

10. Show that Ex. 9 reduces, *in plano*, to Salmon's Theorem.

[We have

$$\frac{AM}{BN} = \frac{PB'}{PA'} = \frac{PA}{PB},$$

since $PA \,.\, PA = PB \,.\, PB'$.]

11. If, in Ex. 9, the point B be on the circle, show that

$$\frac{\sin AM}{\sin BN} = \frac{\cos PA}{\sin PA'} \tan r = \text{constant}.$$

Hence the sine of the perpendicular from any fixed point (A), *on a variable tangent to a small circle, bears a constant ratio to the sine of the perpendicular from the point of contact* (B) *of the tangent, on the polar of the fixed point* (A), *with respect to the circle.*

12. From the vertices of a polygon circumscribed to a small circle perpendiculars are drawn to any variable tangent to it; prove that the continued product of their sines bears a constant ratio to the continued product of the sines of the perpendiculars from the point of contact of the variable tangent on—

(1) The sides of the inscribed polygon formed by joining the points of contact of the circumscribed polygon.

(2) The sides of the circumscribed polygon.

[Apply Ex. 11, and Art. 148, *Cor.* 5].

13. In the case of a polygon of an even number of sides circumscribed to a small circle; prove that the continued product of the sines of the perpendiculars, from the even vertices on any variable tangent, bears a constant ratio to the product of the sines of the perpendiculars from the odd vertices on it; and, if $\delta_1, \delta_2, \delta_3 \ldots \delta_{2n}$ be the angular distances of the vertices from the pole of the small circle, this constant ratio is

$$\frac{\sin \delta_2 \sin \delta_4 \ldots \sin \delta_{2n}}{\sin \delta_1 \sin \delta_3 \ldots \sin \delta_{2n-1}}.$$

[Apply Ex. 11, and Art. 148].

14. If $\delta_1, \delta_2, \ldots \delta_n$ be the angular distances of the pole of the circle in Ex. 12 from the vertices of the circumscribed polygon, and $d_1, d_2 \ldots d_n$ its angular distances from the sides of the inscribed polygon; show the ratios (1) and (2) of that example are, respectively—

$$(1) \qquad \sqrt{\left\{\frac{\sin 2\delta_1 \sin 2\delta_2 \ldots \sin 2\delta_n}{\sin 2d_1 \sin 2d_2 \ldots \sin 2d_n}\right\}}.$$

$$(2) \qquad \frac{\sin \delta_1 \sin \delta_2 \sin \delta_3 \ldots \sin \delta_n}{\sin^n r}.$$

15. If two polygons be reciprocal polars to each other, with respect to a circle (*i.e.* such that each vertex of either is the pole of a side of the other); then the product of the sines of the angular distances of any point on the circle, from the n sides of either, bears a constant ratio to the product of the sines of the angular distances of the vertices of the other, from the tangent at that point.

152. **Theorem.**—*If A and B be two points on a sphere, and if the polar of A passes through B; then the polar of B will pass through A.*

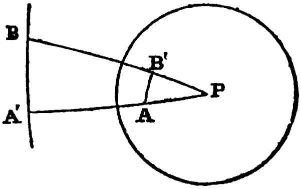

Fig. 59.

Let $A'B$ (fig. 59) be the polar of A passing through B. Join PB, and draw AB' perpendicular to PB. Then, since the angles at A' and B' are right, we have

$$\tan PA' = \tan PB \cos P, \text{ and } \tan PB' = \tan PA \cos P.$$

Therefore

$$\tan PB \,.\, \tan PB' = \tan PA \,.\, \tan PA' = \tan^2 r;$$

and hence AB' is the polar of B.

Examples.

1. If a point moves along a given great circle, its polar turns round a fixed point, viz. the pole of the given great circle.

2. If two points A and B be given, and if their polars intersect in C; then AB is the polar of C.

[For, by this Article, since the polar of A passes through C, the polar of C must pass through A, and similarly it must pass through B].

Hence the arc of connexion of two points is the polar of the intersection of the polars of the points. And again, *the intersection of the polars of two points is the pole of the arc of connexion of the points.*

3. If through any point a secant be drawn cutting a small circle, the tangents at the points where it meets the circle intersect on the polar of the point.

4. If four points A, B, C, D lie on a great circle, their anharmonic ratio is equal to the anharmonic ratio of their four polars.

[For the polars pass through a point, viz. the pole of the great circle, and they are perpendiculars from this point to the arcs joining P, the pole of the small circle to A, B, C, D. Therefore, &c., by Art. 137, *Cor.* 4].

Hence, *if four points form a harmonic row, their four polars form a harmonic pencil.*

5. The locus of the intersection of tangents to a circle, at the extremities of a variable chord, which passes through a fixed point, is the polar of the point.

6. If two triangles are such that each vertex of one is the pole of the corresponding side of the other; then reciprocally each vertex of the latter is the pole of the corresponding side of the former; and the same holds generally for two polygons, or, more generally still, for two curves, such that the tangent at any point of one is the polar of the corresponding point of the other.

7. Show that—

(α) When two points (or great circles) are conjugate, so also are their polars (or poles).

(β) The common conjugate of two points is the point of intersection of their polars, *i. e.* the pole of the great circle joining them.

(γ) Every two conjugate points, with the pole of the great circle joining them, determine a self-reciprocal triangle; and every two conjugate great circles, with the polar of their point of intersection, determine a self-reciprocal triangle.

Definition.—A triangle such that each side is the polar of the opposite vertex is said to be *Self-Reciprocal* or *Self-Conjugate* with respect to the small circle.

153. **Theorem.**—*Any secant drawn through a fixed point across a small circle is cut harmonically by the circle, the point, and the polar of the point with respect to the circle.*

Let a secant through O (fig. 57) meet the circle in M and N, and the polar of O in R. Denote the perpendiculars from M on OT, OT', TT' by p_1, p_2, p, respectively, and the corresponding perpendiculars from N by ϖ_1, ϖ_2, ϖ. Then by Art. 149,

$$\frac{\sin p_1}{\sin \varpi_1} \cdot \frac{\sin p_2}{\sin \varpi_2} = \frac{\sin^2 p}{\sin^2 \varpi}. \qquad (1)$$

But
$$\frac{\sin p_1}{\sin \varpi_1} = \frac{\sin OM}{\sin ON} = \frac{\sin p_2}{\sin \varpi_2}. \qquad (2)$$

Therefore, by (1),

$$\frac{\sin p}{\sin \varpi} = \frac{\sin p_1}{\sin \varpi_1} = \frac{\sin p_2}{\sin \varpi_2}.$$

That is,

$$\frac{\sin RM}{\sin RN} = \frac{\sin OM}{\sin ON};$$

or the arc MN is cut internally at R, and externally at O in the same ratio of sines; therefore, &c.

Examples.

1. If through a fixed point O a variable secant be drawn, cutting a circle in M and N; and if R be taken, the harmonic conjugate of O with respect to M and N, the locus of R is a great circle, viz. the polar of O.

2. If from any point O tangents be drawn to a small circle, the points of contact T, T', and the points of intersection M, N, of any secant from O, form a harmonic system of points.

[Let X be any point on the circle ; then by Art. 150, $X . TMT'N = T . TMT'N = T . OMRN$, which is harmonic].

3. If the vertices of a quadrilateral lie on a small circle, the extremities of its third diagonal and the intersection of its other diagonals are three points, such that the arc joining any two is the polar of the third, with respect to the circle.

[This follows from the harmonic properties of a complete quadrilateral, discussed in Art. 143.]

4. If four points on a circle form a harmonic system, the arc of connexion of any pair passes through the pole of the arc of connexion of the other pair.

[Apply Ex. 2].

5. If tangents be drawn at the vertices of a quadrilateral, inscribed in a small circle, they form a circumscribed quadrilateral, and intersect in six points which lie in pairs on the sides of the self-conjugate triangle, referred to in Ex. 3.

6. The diagonals of the quadrilaterals, inscribed and circumscribed, in Ex. 5 are concurrent, and form a harmonic pencil.

7. The extremities of the third diagonals of the two quadrilaterals in Ex. 5 lie on a great circle, and form a harmonic row.

8. If two triangles be self-conjugate with respect to a circle, any two sides are divided equianharmonically by the four remaining sides, and any two vertices are subtended equianharmonically by the four remaining vertices.

[Apply Ex. 4, Art. 152].

9. If two triangles be such that the sides of one are the polars of the vertices of the other, they are in perspective.

[Proof as *in plano*, for which see Townsend's *Modern Geometry*, vol. i., Art. 180, or Casey's *Sequel to Euclid*].

10. Any four fixed tangents to a small circle are cut by a variable fifth tangent in four points, whose anharmonic ratio is constant.

[The pencil joining the point of contact of the variable tangent to the points of contact of the fixed tangents has a constant anharmonic ratio; but the arcs forming this pencil are the polars of the points in which the variable tangent cuts the fixed tangents; therefore, &c., by Ex. 4, Art. 152.]

11. Hence any two fixed tangents to a small circle are cut homographically by a variable tangent

12. If a hexagon be described about a circle, the three arcs joining the opposite angular points will be concurrent. (Brianchon's *Theorem.*)

[The arcs joining the points of contact form an inscribed hexagon; and therefore, by Art. 150, Ex. 1, the intersections of its opposite sides are concyclic; but these sides are the polars of the opposite vertices of the circumscribed hexagon; therefore their intersections are the poles of the connectors of its opposite vertices; and since they are concyclic, it follows that their polars, viz., the connectors of the opposite vertices in question, are concurrent.]

13. If a great circle drawn across two small circles meet, one in a pair of conjugate points with respect to the other; then it meets the latter in a pair of conjugate points with respect to the former.

154. **The Polar Circle.**—If the perpendiculars of a triangle meet the opposite sides in points X, Y, Z, and if O be the orthocentre, it is easily shown that

$$\tan OA \tan OX = \tan OB \tan OY = \tan OC \tan OZ.$$

Hence, if a circle be described with O as pole, and an angular radius, ρ, given by the equation

$$\tan^2 \rho = \tan OA \tan OX = \&c.,$$

the triangle ABC will be so situated, with respect to it, that each side is the polar of the opposite vertex. The circle thus constructed is called the *Polar Circle* of the triangle. Thus any triangle is self-conjugate with respect to its polar circle.

As the parts of a triangle may each have any value between zero and π, the polar circle may be real or imaginary, whether the orthocentre lies inside or outside the triangle. The expression of the following Article, for its radius, shows that it is real or imaginary according as an odd or an even number of the parts of the triangle exceed 90°.

Examples.

1. The polar circle of any triangle is also the polar circle of its three colunars.

2. The vertices A, B, C, of the triangle ABC, are the spherical centres of the polar circles of the triangles BOC, COA, AOB.

3. The polar circle of the triangle ABC, and the polar circles of BOC, COA, AOB, cut each other orthogonally.

[For BY is the polar of A, with respect to the circle round C; and also the polar of C, with respect to the circle round A. Therefore, &c., by Art. 151, Ex. 8.]

4. The angular radius of the polar circle of any triangle is the complement of the angular radius of the polar circle of its polar triangle, and the latter circle is identical with the polar circles of the polar triangles of the colunars of the given triangle.

155. **Problem.**—*To find the angular radius of the polar circle of a given triangle.*

Let ABC be a given triangle (figs. 60, 61), X, Y, Z the feet of the perpendiculars drawn from the vertices on the opposite sides; O the orthocentre.

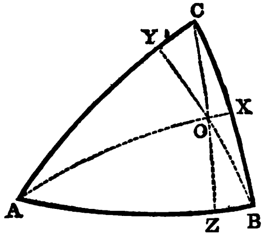

Fig. 60.

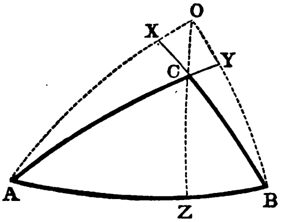

Fig. 61.

Then we have

$$\frac{\tan OC}{\tan CX} = \sec OCX = \frac{\tan a}{\tan CZ}; \qquad (1)$$

[Art. 44.]

and
$$\frac{\tan OZ}{\tan CZ} = \frac{\tan OAZ}{\tan A} = \frac{\tan BX \cos A}{2N}. \qquad (2)$$

Multiplying (1) and (2) together, we obtain

$$\frac{\tan OC \tan OZ}{\tan CX \tan CZ} = \frac{\tan BX \tan a \cos A}{2N \tan CZ}.$$

Hence,

$$\tan OC \tan OZ = \frac{\tan a \tan BX \tan CX \cos A}{2N};$$

or
$$\tan^2 \rho = -\frac{\tan a \tan b \tan c \cos A \cos B \cos C}{2N}.^*$$

[Art. 44 and Art. 36, Ex. (2).]

Examples.

1. The radii ρ_1, ρ_2, ρ_3 of the polar circles of the triangles BOC, COA, and AOB are given by the equations

$$\tan^2 \rho_1 = \tan b \tan c \cos A,$$
$$\tan^2 \rho_2 = \tan c \tan a \cos B,$$
$$\tan^2 \rho_3 = \tan a \tan b \cos C.$$

[For $\tan^2 \rho_1 = \tan AY \tan AC$ (fig. 60) $= \tan AC \tan AB \cos A$.]

2. Prove that $\cos^2 \rho_1 = \frac{\cos b \cos c}{\cos a}$, with similar relations for $\cos^2 \rho_2$ and $\cos^2 \rho_3$.

[$\sec^2 \rho_1 = 1 + \tan^2 \rho_1 = 1 + \tan b \tan c \cos A$ (Ex. 1)

$$= \frac{\cos b \cos c + \sin b \sin c \cos A}{\cos b \cos c} = \frac{\cos a}{\cos b \cos c};$$ therefore, &c.]

3. Prove that

$$\cos a \cos \rho_1 = \cos b \cos \rho_2 = \cos c \cos \rho_3 = (\cos a \cos b \cos c)^{\frac{1}{2}}.$$

---

[*] The negative sign is attached, for when O is inside the triangle (fig. 60), OC and OZ are measured in opposite directions from O; and when O is outside the triangle (fig. 61), we have $\tan CX = -\tan b \cos C$.

4. The polar circles, ρ_1, ρ_2, ρ_3 intersect at right angles.

[From Ex. 3, $\cos a \cos b \cos \rho_1 \cos \rho_2 = \cos a \cos b \cos c$;
hence $\cos c = \cos \rho_1 \cos \rho_2$; and since c is the arc joining the poles of one pair of circles whose radii are ρ_1 and ρ_2; therefore, &c.]

5. Prove the formulæ

$$\tan^2 \rho_2 + \tan^2 \rho_3 + \tan^2 \rho_2 \tan^2 \rho_3 = \tan^2 a,$$
$$\tan^2 \rho_3 + \tan^2 \rho_1 + \tan^2 \rho_3 \tan^2 \rho_1 = \tan^2 b,$$
$$\tan^2 \rho_1 + \tan^2 \rho_2 + \tan^2 \rho_1 \tan^2 \rho_2 = \tan^2 c.$$

[From Ex. 4 we have $\sec^2 a = \sec^2 \rho_2 \sec^2 \rho_3$;

hence $1 + \tan^2 a = (1 + \tan^2 \rho_2)(1 + \tan^2 \rho_3)$.

Or thus:

$\tan^2 \rho_2 = \tan a \tan BX$, and $\tan^2 \rho_3 = \tan a \tan CX$; also $\tan a = \tan(BX + CX)$; therefore, &c.

6. Prove that

$$\frac{\tan^2 \rho_1 \tan^2 \rho_2 \tan^2 \rho_3}{\tan^2 \rho} = -2N \tan a \tan b \tan c.$$

7. The cosine of the distance between two conjugate points is equal to the product of the cosines of the tangents drawn from the points to the small circle.

[A and B are conjugate points, with respect to the polar circle of the triangle, and the tangents from them to it are ρ_1 and ρ_2; and by Ex. 4 we have $\cos c = \cos \rho_1 \cos \rho_2$.
For analogue, *in plano*, see Townsend's *Modern Geometry*, Art. 167.]

8. If t denote the tangent from a point midway between two conjugate points, the tangents from which are t_1 and t_2, prove that

$$\tan^2 t = \frac{\sin^2 t_1 + \sin^2 t_2}{(\cos t_1 + \cos t_2)^2};$$

and state the analogue *in plano*.

[Let M be the middle point of AB; then, by Art. 42,

$$\cos AO + \cos BO = 2 \cos AM \cos OM.$$

Therefore $\cos t_1 + \cos t_2 = 2 \cos \tfrac{1}{2} AB \cos t$;

or

$$\frac{(\cos t_1 + \cos t_2)^2}{2 \cos^2 t} = 2 \cos^2 \tfrac{1}{2} AB = 1 + \cos AB = 1 + \cos t_1 \cos t_2; \text{ therefore, \&c.]}$$

Analogue *in plano* is $4t^2 = t_1^2 + t_2^2$; or $t = \frac{1}{2} AB$.

9. If A and B are two conjugate points, Z the polar centre of AB, and t_3 the tangent from Z to the polar circle, prove that

$$\sin^2 t_3 = \tan AZ \tan BZ.$$

[For (fig. 60) $\quad \tan AZ = \tan ACZ \sin CZ;$

and $\quad \tan BZ = \tan BOZ \sin OZ.$

Therefore

$$\tan AZ \tan BZ = \frac{\sin CZ \sin OZ}{\cos CO} = \frac{\sin CZ \sin OZ}{\cos (OZ \sim CZ)} = \frac{1}{\cot CZ \cot OZ + 1},$$

which, by Art. 151, $\quad = \dfrac{1}{1 + \cot^2 t_3} = \sin^2 t_3.$]

SECTION III.

Coaxal Circles, Radical Axis, and Radical Centre on the Sphere.

156. **Radical Axis.**—*To find the locus of a point, such that arcs drawn from it to touch two small circles shall be equal.*

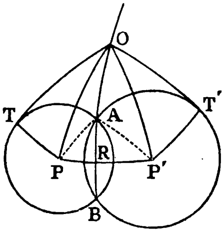

Fig. 62.

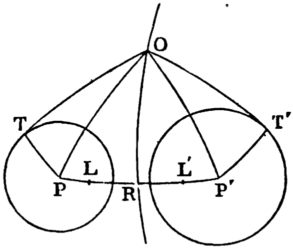

Fig. 63.

Let O be any position of the point (figs. 62, 63), P and P' the poles of the small circles, r and r' their angular radii.

Then $\cos OT = \cos OP \div \cos r$;

also $\cos OT' = \cos OP' \div \cos r'$.

Therefore, if $OT = OT'$,

we have
$$\frac{\cos OP}{\cos OP'} = \frac{\cos r}{\cos r'}.$$

Hence in the triangle POP' we are given the base PP', and the ratio of the cosines of its sides; and therefore the locus of its vertex, O, is a great circle, perpendicular to the base, at a point R, such that
$$\frac{\cos OP}{\cos OP'} = \frac{\cos RP}{\cos RP'} = \frac{\cos r}{\cos r'}.$$
(Art. 56, Ex. 11.)

This circle (Art. 43) obviously coincides with the arc joining the points of intersection of the given circles in case they cut each other, and with the tangent at their point of contact in case they touch. It may also be regarded as the arc joining their imaginary points of intersection if they do not cut each other in real points (fig. 63).

The great circle, OR, from any point of which the tangents to two small circles are equal, is called the *Radical Axis* of the circles; and a system of circles having a common radical axis is termed a *Coaxal System.*

Thus the radical axis of two circles is the great circle joining their points of intersection (real or imaginary); and a coaxal system of circles is a system having two common points (real or imaginary).

Cor. 1.—A circle having any point, O, on the radical axis for pole, and one of the equal tangents, OT, for spherical radius, will cut both the given circles orthogonally; and conversely, every circle cutting both circles orthogonally will have its pole on the radical axis.

Cor. 2.—The radical axis bisects each of the common tangents to two circles.

157. **Radical Centre.**—*The radical axes of three given small circles, taken in pairs, are concurrent.*

By the application of Art. 43 it follows exactly as *in plano*, that the arc joining the intersection of the radical axes of two of the pairs to one of the common points of the third pair must pass through their other common point; therefore, &c.

The point of concurrence of the radical axes of three circles taken in pairs is called their *Radical Centre*, and it lies either within all three or without all three.

Cor. 1.—Hence we can describe a circle to cut any three circles orthogonally, its pole being their radical centre.

Cor. 2.—If any circle cut a coaxal system of circles, the spherical chords of intersection of it, with each of the circles, will all pass through a point on the radical axis of the system.

Example.

The three arcs drawn through the vertices of a triangle bisecting its area are concurrent.

[Let the arcs meet the sides a, b, c in points X, Y, Z, respectively. Then, by Art. 101, the points X and Y lie on a small circle passing through the points A', B', diametrically opposite to A and B. Similarly, Y and Z lie on a circle passing through B' and C', and Z and X lie on a circle passing through C' and A'. Thus we have three small circles intersecting in X, Y, Z, A', B', C'; and their radical axes being $A'X$, $B'Y$, $C'Z$, are the bisectors of the area of ABC.]

158. **Limiting Points.**—*Any circle cutting two given circles orthogonally passes through two fixed points on the arc joining their poles.*

Let O (fig. 63) be the pole of a circle cutting two circles orthogonally, and let it intersect PP' in L and L'. Then we have already seen that O is on the radical axis, and $OT = OL$. Therefore

$$\frac{\cos RP}{\cos RL} = \frac{\cos OP}{\cos OT} = \cos r.$$

Hence the distance RL ($= RL'$) is independent of the position of O on the radical axis, and is equal to the tangent from R to either of the circles.

The points L and L', through which all circles orthogonal to both the given circles pass, are called their *Limiting Points*, since they may be considered as a pair of infinitely small circles coaxal with the given ones.

Cor. 1.—If a system of circles be cut orthogonally by two circles, it is a coaxal system.

Cor. 2.—The system of circles orthogonal to a given coaxal system is also a coaxal system, and the two systems are related as follows:—

(α) Every circle of one system cuts orthogonally every circle of the other system.

(β) Every circle of one system passes through the limiting points (real or imaginary) of the other system.

(γ) The limiting points of one system are the common points of the other.

(δ) If the limiting points of one system are real, the limiting points of the other are imaginary.

(ϵ) The poles of either system lie on the radical axis of the other.

159. **Fundamental Theorem.**—*If from any point tangents be drawn to two small circles, the difference of their cosines bears a constant ratio to the sine of the perpendicular from the point to the radical axis.*

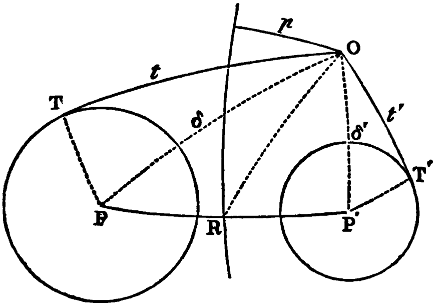

Fig. 64.

Let P and P' (fig. 64) be the poles of the given circles, r and r' their angular radii, δ and δ' the angular distances OP and OP'. Then the triangles ORP and ORP' give us

$$\cos\delta = \cos PR \cos OR - \sin PR \sin p,$$

and

$$\cos\delta' = \cos P'R \cos OR + \sin P'R \sin p,$$

since $\sin p = \sin OR \cos ORP'$.

Multiplying the first of these equations by $\cos r'$, and the second by $\cos r$, and remembering that

$$\cos r \cos P'R = \cos r' \cos PR,$$

we find, by subtracting,

$$\sin p\,(\sin PR \cos r' + \sin P'R \cos r) = \cos r \cos\delta' - \cos r' \cos\delta$$

$$= \cos r \cos r' \left(\frac{\cos\delta'}{\cos r'} - \frac{\cos\delta}{\cos r}\right) = \cos r \cos r'\,(\cos t' - \cos t).$$

But if γ denote half the common chord of the two circles, it follows from Art. 40 (see fig. 62) that

$$\sin PR \cos r' + \sin P'R \cos r = \cos \gamma \sin D,$$

where D (= PP') is the angular distance between the poles of the circles.

Therefore

$$\frac{\cos t' - \cos t}{\sin p} = \frac{\cos \gamma \sin D}{\cos r \cos r'}.$$

160. **Important Case of Fundamental Theorem.**—If, in the foregoing Article, we suppose the point O to be on one of the circles (P', suppose), we have $t' = 0$, and $\cos t' = 1$; therefore it follows that:—*If from any point O on one of two given small circles a tangent t be drawn to the other, and a perpendicular p be let fall on their radical axis, then*

$$\frac{\sin^2 \frac{1}{2} t}{\sin p} = \text{constant} = \frac{\cos \gamma \sin D}{2 \cos r \cos r'}.$$

And conversely—*The locus of a point such that the square of the sine of half the tangent from it to a given small circle bears a constant ratio to the sine of the perpendicular from it on a fixed great circle, is a small circle, which, with the given small circle, has the great circle for radical axis.*

This theorem is of fundamental importance in the theory of coaxal circles on the sphere, many of their most important properties flowing directly from it. The student who is familiar with the corresponding properties in plane geometry will experience no difficulty in applying it to deduce the following corollaries:—

Cor. 1.—If tangents t_1, t_2 be drawn to two circles from any point on a circle coaxal with them; then

$$\sin \tfrac{1}{2} t_1 : \sin \tfrac{1}{2} t_2 = \text{constant}.$$

And conversely—*The locus of a point such that the sines of the halves of the tangents drawn from it to two fixed circles are in a constant ratio is a circle coaxal with the given circles.*

[This may be easily proved directly].

Cor. 2.—The sine of half the tangent from any point to a circle of a coaxal system varies as the sine of half its angular distance from a limiting point.

Cor. 3.—If a chord of one circle touch another, the ratio of the sines of the halves of its segments is equal to the ratio of the sines of the halves of the chords joining its extremities to a limiting point.

Cor. 4.—If the circumcircle of a triangle ABC touch a given circle to which the tangents from A, B, C are t_1, t_2, t_3; then

$$\sin \tfrac{1}{2} a \sin \tfrac{1}{2} t_1 + \sin \tfrac{1}{2} b \sin \tfrac{1}{2} t_2 + \sin \tfrac{1}{2} c \sin \tfrac{1}{2} t_3 = 0.$$

[The limiting points coincide with the point of contact. Hence, applying *Ptolemy's Theorem* on the sphere (Art. 25, Ex. 6) to the quadrilateral formed by A, B, C and the point of contact, we obtain the required result by the aid of *Cor.* 2].

Examples.

1. If r_1, r_2, r_3 be the angular radii of three coaxal circles; D_{23}, D_{31}, D_{12} the angular distances between their poles, prove the relation

$$\sin D_{23} \cos r_1 + \sin D_{31} \cos r_2 + \sin D_{12} \cos r_3 = 0.$$

[Take any point, O, on the radical axis, and let P_1, P_2, P_3 be the poles of the circles; then, by Art. 40,

$$\sin D_{23} \cos OP_1 + \sin D_{31} \cos OP_2 + \sin D_{12} \cos OP_3 = 0;$$

and since the tangents from O to the circles are equal, this equation reduces to that in question.]

Note.—The above equation may be regarded as a criterion of the coaxality of three circles.

2. If a circle of radius, r, cut three coaxal circles at angles α, β, γ, show that

$$\sin D_{23} \sin r_1 \cos \alpha + \sin D_{31} \sin r_2 \cos \beta + \sin D_{12} \sin r_3 \cos \gamma = 0.$$

[Let D_1, D_2, D_3 be the angular distances of the pole of this circle from the poles of the given circle. Then

$$\begin{aligned}\cos D_1 &= \cos r \cos r_1 + \sin r \sin r_1 \cos \alpha,\\ \cos D_2 &= \cos r \cos r_2 + \sin r \sin r_2 \cos \beta,\\ \cos D_3 &= \cos r \cos r_3 + \sin r \sin r_3 \cos \gamma.\end{aligned}$$

Multiplying these equations by $\sin D_{23}$, $\sin D_{31}$, $\sin D_{12}$, respectively, and remembering that $\Sigma \sin D_{23} \cos D_1 = 0$ (Art. 40), and that by the foregoing Example, $\Sigma \sin D_{23} \cos r_1 = 0$, we have, by addition,

$$\Sigma \sin D_{23} \sin r_1 \cos \alpha = 0.]$$

3. Hence show that a circle cutting two circles at constant angles intersects any circle coaxal with them at a constant angle; and therefore a circle cutting two circles at constant angles touches *two* circles coaxal with them.

4. The products of the tangents of the halves of the segments of an arc, drawn from any point of the radical axis cutting any circle of a coaxal system, is the same for all circles of the system (the segments being measured from the point).

[Apply Art. 43.]

5. If two circles cut three given circles, the triangles formed by the common chords (arcs of great circles) of each with the three are in perspective, and the centre of perspective is the radical centre of the given circles.

[For the corresponding sides of the two triangles meet in pairs on the radical axis of the two circles.]

6. If a variable circle passes through two fixed points on a sphere, and also cuts another fixed circle, their common chord, that is the arc joining their points of section, passes through a fixed point on the great circle joining the fixed points (Art. 43).

7. Describe a circle such that the radical axes of it and three given circles shall pass through three given points.

[This reduces to Art. 145, Ex. 13.]

8. If from a point, O, on the radical axis of a coaxal system of small circles, a great circle be drawn cutting them in points AA', BB', &c., prove that the middle points of the arcs OA, OA'; OB, OB', &c., will form a system of points in spherical involution, of which O is the centre.

[For we have $\tan\frac{1}{2}OA \tan\frac{1}{2}OA' = \tan\frac{1}{2}OB \tan\frac{1}{2}OB' =$ &c.; hence the mid-points are in involution by Art. 147.]

9. If two circles cut two other circles orthogonally, the radical axis of either pair is the great circle joining the poles of the other pair.

[For each pair passes through the limiting points of the other pair.]

10. If four circles be mutually orthogonal, the six great circles joining their poles in pairs are also the radical axes of them taken in pairs.

11. Hence the vertices, A, B, C, of a triangle, and the orthocentre, O, are the radical centres of the four triads of circles formed by taking the polar circles of the triangles ABC, BOC, COA, AOB, three at a time.

12. The polars, with respect to two circles, of any point on their radical axis intersect on the radical axis.

13. The chords of intersection of either of two small circles, with a system of circles orthogonal to both, pass through the pole of their radical axis with respect to it.

14. If a variable circle intersect two fixed circles of angular radii, r_1 and r_2, at equal angles (or at two angles whose cosines have a constant ratio), and if tangents be drawn to the fixed circles from the pole of the variable circle, and from their points of contact tangents T_1 and T_2 be drawn to the variable circle; show that

$$\frac{\sin^2\frac{1}{2}T_1}{\sin^2\frac{1}{2}T_2} = \text{constant.}$$

[Let ρ be the angular radius of the variable circle, δ_1 and δ_2 the distances of its pole from the poles of given circles, t_1 and t_2 the tangents from its pole to them; then

$$\cos\delta_1 = \cos\rho\cos r_1 + \sin\rho\sin r_1\cos\alpha.$$

Therefore $$\cos t_1 = \cos\rho + \sin\rho\tan r_1\cos\alpha;$$

and $$\cos T_1 = 1 + \tan\rho\tan r_1\cos\alpha.$$

Similarly $\cos T_2 = 1 + \tan\rho\tan r_2\cos\beta$. Therefore, &c.]

15. If a circle touch two fixed circles, the ratio of the sines of the halves of the tangents to it from their limiting points is constant.

SECTION IV.

Centres and Axes of Similitude on the Sphere.

161. **Problem.**—*To draw a great circle touching two small circles on a sphere.*

Let T_1T_2 (fig. 65) be a great circle touching two small circles at T_1 and T_2, respectively. Let P_1 and P_2 be the poles of the small circles; then P_1T_1, and P_2T_2 meet in P, the pole of T_1T_2. Now, the sides PP_1 and PP_2 of the triangle PP_1P_2 are $90 - r_1$, and $90 - r_2$, where r_1 and r_2 are the radii of the circles; and therefore if the small circles be given, the point P is at once determined, and the points T_1 and T_2, where PP_1 and PP_2 produced meet the circles, give the points of contact of the required common tangent.

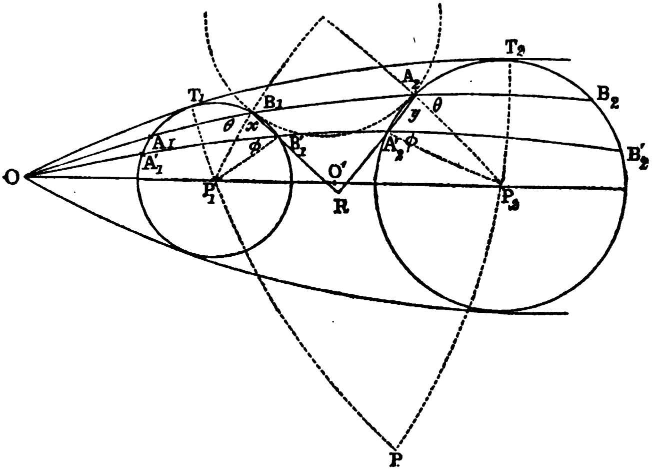

Fig. 65.

Cor.—It is always possible to draw a pair of direct common tangents, except when one circle is entirely within the other; and a pair of transverse common tangents when one circle is entirely outside the other.

162. **Centres of Similitude.**—The point of intersection of the direct common tangents to two small circles is called their *External Centre of Similitude*, and the point of intersection of the transverse common tangents their *Internal Centre of Similitude.*

It is clear that the centres of similitude of two circles lie on the great circle joining their poles. For OP_1 bisects the angle between the direct common tangents, and so also does OP_2. Therefore O lies on P_1P_2; similarly O' lies on P_1P_2.

Cor.—The point of contact of two circles which touch externally is their internal centre of similitude, and the point of contact of two which touch internally is their external centre of similitude.

163. **Theorem.**—*The centres of similitude divide the arc joining the poles of the circles internally and externally into segments, whose sines are in the ratio of the sines of the spherical radii of the circles.*

For (fig. 65) if r_1 and r_2 denote the spherical radii,

$$\frac{\sin r_1}{\sin OP_1} = \frac{\sin r_2}{\sin OP_2} = \sin P_1OT_1,$$

and similarly for the internal centre; therefore, &c.

Cor. 1.—*Any great circle through a centre of similitude makes equal angles with the radii to the points where it cuts the small circles.*

[For (see fig. 65), $\sin OP_1 \sin P_1OB_1 = \sin r_1 \sin OB_1P_1$,

and $\sin OP_2 \sin P_1OB_1 = \sin r_2 \sin P_2A_2B_2$.

Therefore, by the above proposition,

$$OB_1P_1 = P_2A_2B_2.$$

Hence we have

$$P_1A_1B_1 = P_1B_1A_1 = P_2A_2B_2 = P_2B_2A_2.]$$

Cor. 2.—Conversely—*If an arc of a great circle cuts two small circles, and makes equal angles with the radii to the points of section, it passes through a centre of similitude.*

Cor. 3.—Hence, *if a small circle touch two other small circles, the arc joining the points of contact passes through a centre of similitude.*

Cor. 4.—The centres of similitude and the poles of the given circles form a harmonic row.

[Art. 136].

Definition.—Of the four points A_1, B_1, A_2, B_2 (fig. 65), in which any great circle through a centre of similitude of two circles meets them, the points A_1 and A_2, or B_1 and B_2, are said to correspond directly, and the points A_1 and B_2, or B_1 and A_2, are said to correspond inversely.

164. **Theorem.**—*If two great circles be drawn through a centre of similitude of two small circles, any pair of the points in which they intersect the small circles are concyclic with their inversely corresponding points.*

Consider the quadrilateral (fig. 65) $B_1A_2A_2'B_1'$ formed by the points B_1, B_1', and their inversely corresponding points A_2, A_2'. Denote each of the equal angles of the triangle $P_1B_1B_1'$ by x, and those of $P_2A_2A_2'$ by y, and let θ denote the equal angles P_1B_1O and $P_2A_2B_2$, and ϕ the angles $P_1B_1'O$ and $P_2A_2'B_2'$. Then, for the angles of the quadrilateral, we have

$$RB_1A_2 = \pi - x - \theta, \text{ and } B_1B_1'A_2' = \pi - (x - \phi).$$

Therefore

$$B_1' - B_1 = \theta + \phi = A_2' - A_2;$$

or

$$A_2 + B_1' = A_2' + B_1;$$

and therefore the quadrilateral $B_1A_2A_2'B_1'$ is such that the sum of one pair of opposite angles is equal to the sum of the other pair; that is, it is inscribable in a small circle. [Art. 19 (4).]

Cor. 1.—Hence,

$$\tan \tfrac{1}{2} OB_1 \tan \tfrac{1}{2} OA_2 = \tan \tfrac{1}{2} OB_1' \tan \tfrac{1}{2} OA_2' = \tan \tfrac{1}{2} OT_1 \tan \tfrac{1}{2} OT_2.$$

Therefore—*The product of the tangents of the halves of the arcs intercepted between either centre of similitude and a pair of points inversely corresponding is constant.*

Cor. 2.—Combining *Cor.* 1 with Art. 43, we find

$$\frac{\tan \frac{1}{2} OA_1}{\tan \frac{1}{2} OA_2} = \frac{\tan \frac{1}{2} OB_1}{\tan \frac{1}{2} OB_2} = \frac{\tan \frac{1}{2} OT_1}{\tan \frac{1}{2} OT_2}.$$

Therefore—*The tangents of the halves of the arcs between either centre of similitude and a pair of corresponding points are in a constant ratio.*

Cor. 3—If B_1B_1', and A_2A_2' be produced to meet at R, we have

$$\tan \tfrac{1}{2} RB_1 \tan \tfrac{1}{2} RB_1' = \tan \tfrac{1}{2} RA_2 \tan \tfrac{1}{2} RA_2',$$

and therefore R is on the radical axis of the two circles.

Therefore—*The arc joining a pair of points meets the arc* *joining their inversely corresponding points on the radical axis.*

Cor. 4.—And hence also *tangents to two small circles at two inversely corresponding points meet on their radical axis,* a result which also appears obvious, since a pair of inversely corresponding points may (by Art. 163, *Cor.* 3) be considered as the points of contact of a small circle, and this *Cor.* merely asserts that the radical axes of the three are concurrent.

Cor. 5.—The four points of contact of the direct common tangents are concyclic, and so also are the four points of contact of the transverse common tangents.

Cor. 6.—If a circle touch two given circles, the points of contact are concyclic with the limiting points.

[For the points of contact are concyclic with the points of contact of a common tangent; but these latter points are concyclic with the limiting points, all four being equidistant from the middle point of the common tangent; therefore, &c., by Art. 43].

165. **Axes of Similitude.**—*The six centres of similitude of three small circles taken in pairs lie, three by three, on four great circles, called axes of similitude.*

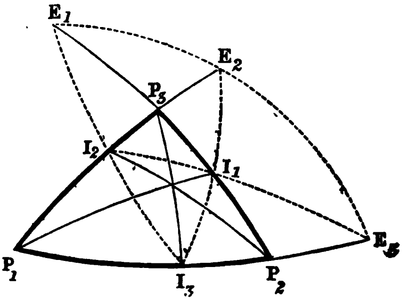

Fig. 66.

Let P_1, P_2, P_3 be the poles of the small circles. I_1, I_2, I_3 their internal centres of similitude. Since the internal centres divide the sides of the triangle P_1, P_2, P_3 into segments whose sines are in the ratio of the sines of the radii, it appears at once (Art. 122) that the points I_1, I_2, I_3 connect concurrently with P_1, P_2, P_3. Again, the arc I_1I_2 meets P_1P_2 in a point E_3, such that

$$\frac{\sin E_3 P_1}{\sin E_3 P_2} = \frac{\sin r_1}{\sin r_2}.$$

[Art. 125]

Therefore E_3 is the external centre of P_1 and P_2. Similarly E_1 and E_2 are the external centres of the other pairs, and they are concyclic. However, this last result may be obtained at once from the criterion of Art. 122.

The concyclic triads are therefore the three external centres, and each external centre, with a pair of internal centres; and the quadrilateral formed by these four great circles is such that the triangle formed by the poles of the small circles is its diagonal triangle.

Cor.—If a circle touch two others, the spherical chord of contact passes through a centre of similitude of the two.

Examples.

1. If an infinite number of small circles touch two others, the radical axis of each pair that can be taken passes through the external centre of similitude, when the two contacts are of the same kind, and through the internal centre when they are of different kinds.

[Since the points of contact are inversely corresponding points, the product of the tangents of the halves of the arcs between them and a centre of similitude is constant, and the same for any two having the same kind of contact. Therefore the tangents from it to these circles are equal, and hence it is a point on their radical axis.]

2. If a system of circles have the same kind of contact with two given ones, they are all cut orthogonally by a circle having its pole at a centre of similitude.

[Apply Ex. 1.]

3. If two circles touch each other, and also two given circles, either both internally or both externally, the locus of the point of contact is a circle having its pole at the external centre of similitude of the given circles.

4. The envelope of a circle touching a given one, and cutting another orthogonally, is a small circle.

5. Great circles touching a pair of small circles at a pair of inversely corresponding points make equal angles with the great circle joining the points.

[Art. 163, *Cor*. 1].

6. Tangents at corresponding points are equally inclined to the secant, and hence tangents at non-corresponding points are equal, and therefore meet on the radical axis.

7. Describe a circle through a given point on a sphere to touch two given circles.

[Join the given point A to a centre of similitude O, and the point B, where it meets the required circle again, is determined by the fact that $\tan \frac{1}{2} OA \tan \frac{1}{2} OB$ is given. Hence the problem is reduced to describing a circle through two points to touch a given one, which, by Art. 43, is reduced to describing a circle through three given points.]

8. Describe a small circle touching three given small circles.

[Four pairs may be described as *in plano* by aid of the four axes of similitude.]

9. Each of the pairs of circles which can be drawn to touch three given circles have the radical centre of the given circles for internal centre of similitude, and the radical axis of each pair is a corresponding axis of similitude.

10. A great circle drawn from the radical centre perpendicular to the axis of similitude, corresponding to a pair of touching circles, passes through their poles.

11. The circle orthogonal to three given circles belongs to each of the systems of circles coaxal respectively with each of the four pairs of circles which can be drawn touching the given circles.

SECTION V.

Spherical Inversion.

166. **Inverse Points.**—In plane geometry, when two points are so situated on a diameter of a circle that the rectangle under their distances from its centre is equal to the square of its radius, each point is termed the inverse of the other with respect to the circle. So on the sphere, any two points A and A' (fig. 67) so situated on any spherical radius of a circle, that

$$\tan \tfrac{1}{2} OA \tan \tfrac{1}{2} OA' = \tan^2 \tfrac{1}{2} a$$

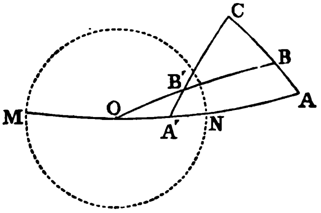

Fig. 67.

(where O is the pole of the circle, and a its angular radius), may be termed *Inverse Points*, each of the other, with respect to the circle on the sphere. The circle is called, in relation to the points, the *Circle of Inversion*, its angular radius the *Radius of Inversion*, and its spherical centre the *Centre of Inversion*, or *Origin*.

More generally, any three points situated on a great circle are such that any pair of them are inverse points with respect to a circle, real or imaginary, having the third for pole—the circle being real or imaginary, according as the points lie on the same side or opposite sides of its pole.

Cor. 1.—The inverse of the centre of inversion is 180° distant from it, *i. e.* the diametrically opposite point.

Cor. 2.—Any point on the sphere has one and only one inverse with respect to any given circle.

Cor. 3.—Any point on the circle of inversion is its own inverse.

Cor. 4.—*Any two points A, B, and their inverses A', B' are concyclic.*

[For $\tan\frac{1}{2}OA \tan\frac{1}{2}OA' = \tan\frac{1}{2}OB \tan\frac{1}{2}OB'$].

Examples.

1. Any circle passing through a pair of inverse points intersects the circle of inversion orthogonally.

2. The limiting points of a system of coaxal circles are inverse points with respect to every circle of the system.

[For any circle passing through them intersects every circle of the system orthogonally.]

3. If two circles intersect orthogonally, each meets every spherical diameter of the other, in a pair of inverse points with respect to it.

4. Any two circles have a common pair of inverse points, viz. their limiting points.

5. If a circle passes through a pair of inverse points with respect to one circle, and also through a pair of inverse points with respect to another; it also passes through their common pair of inverse points.

[For, by Ex. 1, it intersects both circles orthogonally.]

6. If A and A' be a pair of inverse points on a diameter MN of a small circle whose pole is O, show that

$$\frac{\sin OA}{\sin OA'} = \frac{\sin^2\frac{1}{2}MA}{\sin^2\frac{1}{2}MA'} = \frac{\sin^2\frac{1}{2}NA}{\sin^2\frac{1}{2}NA'}.$$

[We have

$$\frac{\tan\frac{1}{2}OA}{\tan\frac{1}{2}OM} = \frac{\tan\frac{1}{2}OM}{\tan\frac{1}{2}OA'} = \frac{\tan\frac{1}{2}OA \pm \tan\frac{1}{2}OM}{\tan\frac{1}{2}OM \pm \tan\frac{1}{2}OA'};$$

therefore

$$\frac{\tan\frac{1}{2}OA}{\tan\frac{1}{2}OA'} = \left(\frac{\tan\frac{1}{2}OA \pm \tan\frac{1}{2}OM}{\tan\frac{1}{2}OM \pm \tan\frac{1}{2}OA'}\right)^2; \text{ \&c.}]$$

NOTE—The analogue *in plano* is

$$\frac{OA}{OA'} = \left(\frac{MA}{MA'}\right)^2 = \left(\frac{NA}{NA'}\right)^2$$

7. If X be any variable point on a small circle, A and A' two inverse points with respect to it, O the pole of the circle; show that

$$\frac{\sin OA}{\sin OA'} = \frac{1 - \cos AX}{1 - \cos A'X} = \left(\frac{\sin\frac{1}{2}AX}{\sin\frac{1}{2}A'X}\right)^2.$$

[We have

$$\sin AA' \cos OX + \sin OA' \cos AX = \sin OA \cos A'X. \qquad (1)$$

But

$$\tan\tfrac{1}{2}OA \tan\tfrac{1}{2}OA' = \tan^2\tfrac{1}{2}OX.$$

Therefore

$$\frac{\cos\frac{1}{2}(OA + OA')}{\cos\frac{1}{2}(OA - OA')} = \frac{1 - \tan^2\frac{1}{2}OX}{1 + \tan^2\frac{1}{2}OX} = \cos OX,$$

which, when substituted for $\cos OX$ in (1), gives at once the required result.]

8. Find the locus of a point such that the sines of the halves of its angular distances from two fixed points have a constant ratio. (Cf. Exam. Paper, VI., Ex. 2.)

[It follows from Ex. 7 that the locus is a circle having the fixed points for a pair of inverse points, as is also evident from Art. 160.]

9. Hence determine a point, the ratios of the sines of the halves of whose angular distances from three fixed points shall be given.

167. **Inverse Figures.**—Any figure (F) on a sphere may be transformed into another (F') by changing all its points into their inverses with respect to any circle chosen arbitrarily. This process of transformation is termed

Inversion, and the figures F and F' are said to be *Inverse Figures*. Thus, if the point A (fig. 67) describe any curve, its inverse A' will describe another curve, viz. the inverse of that described by A.

Any great circle drawn through the centre of inversion meets the curves F and F' in pairs of inverse points; the points in which it meets one curve being the inverses of the *corresponding points* in which it meets the other. It is thus clear that the great circle joining two corresponding points passes through the centre of inversion. The following corollaries will be evident from the definitions.

Cor. 1.—The points of intersection of a curve and its inverse lie on the circle of inversion.

Cor. 2.—The tangents from the centre of inversion to any curve are also tangents to its inverse at the corresponding points.

Cor. 3.—If two curves touch each other, their inverses will also touch each other at the corresponding points.

[For at each point of contact the curves have two consecutive points common, and therefore their inverses have also two consecutive points common at each of the corresponding points.]

168. **Theorem.**—*The tangents to a curve and its inverse at two corresponding points are equally inclined to the great circle joining the points.*

Let A and B (fig. 67) be two consecutive points on any curve; then A' and B', their inverses, will be consecutive points on the inverse curve, and the great circles AB and $A'B'$ will be tangents to the two curves at A and A'; but the four points A, B, A', B' are concyclic (Art. 166, *Cor.* 4), and therefore AB and $A'B'$ are tangents to a small circle, and AA' is their chord of contact. Therefore AB and $A'B'$ are equally inclined to AA'.

Cor.—The angle of intersection of two spherical curves is equal to the angle of intersection of their inverse curves at the corresponding point.

[For, by the foregoing theorem, if A be a point of intersection of two curves, and A' the corresponding point of intersection of their inverses; then the tangents to the curves at A, and the tangents at A' to their inverses make the same angles with AA', and therefore the angle between the former pair of tangents is equal to the angle between the latter pair.

Cor. 3, Art. 167, is obviously a particular case of the above.]

169. **Theorem.**—*The inverse of any circle on a sphere is a circle.*

This theorem has been already anticipated in Art. 164, *Cor.* 1, from which it follows at once that any pair of circles are inverses, each of the other, with respect to a centre of similitude as origin and radius of inversion (a) given by the equation (fig. 65)

$$\tan^2 \tfrac{1}{2} a = \tan \tfrac{1}{2} OT_1 \tan \tfrac{1}{2} OT_2.$$

When the radius of inversion is given, and the origin fixed, this equation determines the points of contact of the inverse circle with the common tangents, and therefore fixes it in magnitude and position. The following is a direct investigation of the theorem, and of the position and radius of the inverse circle.

Let O (fig. 68) be the centre of inversion, P the pole of the circle, A any point on it, and A' its inverse. Draw

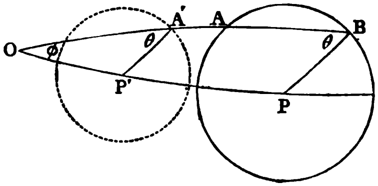

Fig. 68.

the arc $A'P'$, making $OA'P' = OBP = \theta$, and denote OP and OP' by δ and δ', PA and $P'A'$ by r and r', respectively; then if $AOP = \phi$, we have

$$\frac{\sin\delta}{\sin r} = \frac{\sin\delta'}{\sin r'} = \frac{\sin\theta}{\sin\phi};$$

or

$$\frac{\tan\frac{1}{2}(\delta'+r')}{\tan\frac{1}{2}(\delta'-r')} = \frac{\tan\frac{1}{2}(\delta+r)}{\tan\frac{1}{2}(\delta-r)}. \qquad (1)$$

Also, by Napier's Analogies applied to the triangles $OP'A'$ and OPB, we have

$$\frac{\tan\frac{1}{2}(\delta'+r')}{\tan\frac{1}{2}(\delta+r)} = \frac{\tan\frac{1}{2}OA'}{\tan\frac{1}{2}OB}. \qquad (2)$$

But

$$\tan\tfrac{1}{2}OA\tan\tfrac{1}{2}OA' = \tan^2\tfrac{1}{2}a;$$

and

$$\tan\tfrac{1}{2}OA\tan\tfrac{1}{2}OB = \tan\tfrac{1}{2}(\delta+r)\tan\tfrac{1}{2}(\delta-r),$$

by Art. 43. Hence from (2) we find

$$\tan\tfrac{1}{2}(\delta'+r') = \frac{\tan^2\frac{1}{2}a}{\tan\frac{1}{2}(\delta-r)},$$

and therefore, by (1),

$$\tan \tfrac{1}{2} (\delta' - r') = \frac{\tan^2 \frac{1}{2} a}{\tan \frac{1}{2} (\delta + r)}.$$

Hence we have finally, by taking tan (sum), and tan (difference), after a little simple reduction,

$$\tan r' = \frac{\sin^2 a \sin r}{(1 + \cos^2 a) \cos r - 2 \cos a \cos \delta} = \frac{\sin^2 a \tan r}{1 - 2 \cos a \cos \delta + \cos^2 a},$$

and

$$\tan \delta' = \frac{-\sin^2 a \sin \delta}{(1 + \cos^2 a) \cos \delta - 2 \cos a \cos r} = \frac{-\sin^2 a \tan \delta}{1 - 2 \cos a \sec \delta + \cos^2 a}$$

These equations, since they determine r' and δ' in terms of r and δ, show that the point P' is fixed, and that A' describes a circle round it as pole.

Cor. 1.—Any great circle drawn through the centre of inversion intersects a circle and its inverse at equal angles.

[*Cf.* Art. 163, *Cor.* 1, which is only a particular case of Art. 168.]

Cor. 2.—Any circle, its inverse, and the circle of inversion, are coaxal.

Cor. 3.—The centre of inversion is a centre of similitude of the circle and its inverse.

Cor. 4.—Every *great circle* is its own inverse when the centre of inversion is situated on it.

Cor. 5.—When the radius of inversion is 90° (*i. e.* when the circle of inversion is a great circle), every circle is inverted into an equal circle, and therefore *every great circle into a great circle.*

Cor. 6.—When the centre of inversion is on the circle, the inverse circle passes through *the point diametrically opposite to the origin;* and, conversely, when a circle passes through the point diametrically opposite to the origin, *the inverse circle passes through the centre of inversion.*

Cor. 7.—If two circles touch (or intersect orthogonally), their inverses touch (or intersect orthogonally). [Art. 168.]

Hence every spherical diameter of a circle inverts into a circle, cutting the inverse circle orthogonally.

Cor. 8.—Two circles and their inverses are a system of four circles, such that any circle touching any three of them also touches the fourth.

Cor. 9.—Any circle can be inverted into a great circle, [For, if $r' = 90°$, we have $2 \cos a \cos \delta = (1 + \cos^2 a) \cos r$, which determines δ in terms of a and r.]

Cor. 10.—Hence the locus of a point from which, as centre of inversion, a circle is inverted into a great circle (or circle of given angular radius) is a small circle, copolar with the given circle, and its angular radius (δ) is given by the equation

$$\cos \delta = \frac{\cos r}{2 \cos a} (1 + \cos^2 a) = \cos r \, \frac{1 + \tan^4 \frac{1}{2} a}{1 - \tan^4 \frac{1}{2} a}.$$

Remark.—The locus circle coincides with the given circle when it is a great circle, which agrees with *Cor.* 4, or when $a = 0$. The above formula reduces *in plano* to $\delta = r$, or the locus circle coincides with the given circle.

Cor. 11.—Hence any two circles can be inverted into two great circles, viz. from either of the points of intersection of the two locus circles of *Cor.* 10 as centre of inversion.

which is a linear relation between $\cos\delta_1$ and $\cos\delta_2$; and therefore the centre of inversion may lie anywhere on the circle represented by it (Art. 134).

Thus *the locus of a point from which, as centre of inversion, two circles can be inverted into two equal circles, is a small circle.*

Cor. 1.—Any three circles can be inverted into three equal circles; and hence a circle can be described to touch three circles on a sphere, or to intersect three circles at given angles.

Cor. 2.—The circle which is the locus of the centre of inversion, in the foregoing Problem, is coaxal with the two given circles only when they are equal, in which case it is their radical axis.

Cor. 3.—*Any two circles can be inverted into two circles having the tangents of their radii in a constant ratio, and the locus of the origin is a circle.*

Cor. 4.—Any three circles can be inverted into three circles, having the tangents of their radii in any ratio $l : m : n$.

173. **Theorem.**—*The anharmonic ratio of any four points, A, B, C, D, on a circle, is equal to the anharmonic ratio of their four inverses, A', B', C', D', on the inverse circle.*

From O, the centre of inversion, draw a common tangent to the circle and its inverse, and let its points of contact with them be T and T', respectively. Then, since T, A, T', A' are concyclic, TA and $T'A'$ meet on the radical axis of the two circles (cf. Art. 164, *Cor.* 3). Similarly, TB and $T'B'$, TC and $T'C'$, TD and $T'D'$, meet on the radical axis. Hence, by Art. 137, *Cor.* 2, the anharmonic ratios of the pencils $T.ABCD$ and $T'.A'B'C'D'$ are equal. Therefore, &c., by Art. 150.

Cor. 1.—A harmonic system of points on a circle inverts into a harmonic system of points on a circle.

Cor. 2.—If A, B, C, D be any four points on a circle, A', B', C', D' their inverses, then, by Art. 150,

$$\frac{\sin\frac{1}{2}BC\sin\frac{1}{2}AD}{\sin\frac{1}{2}B'C'\sin\frac{1}{2}A'D'}=\frac{\sin\frac{1}{2}CA\sin\frac{1}{2}BD}{\sin\frac{1}{2}C'A'\sin\frac{1}{2}B'D'}$$
$$=\frac{\sin\frac{1}{2}AB\sin\frac{1}{2}CD}{\sin\frac{1}{2}A'B'\sin\frac{1}{2}C'D'}.$$

174. **Theorem.**—*The common tangent, t, and the angular radii, r_1 and r_2, of two circles, are connected with the common tangent, t', and angular radii, r'_1, r'_2 of the inverse circles by the relation*

$$\frac{\sin^2\frac{1}{2}t}{\tan r_1\tan r_2}=\frac{\sin^2\frac{1}{2}t'}{\tan r_1'\tan r_2'}=\sin^2\tfrac{1}{2}a,$$

where a is the angle of intersection of the circles.

Let δ denote the angular distance between the poles of the circles. Then

$$\cos\delta=\cos r_1\cos r_2+\sin r_1\sin r_2\cos a.$$

Also $$\cos\delta=\sin r_1\sin r_2+\cos r_1\cos r_2\cos t.$$

Therefore $$1-\cos t=\tan r_1\tan r_2(1-\cos a);$$

or

$$\frac{\sin^2\frac{1}{2}t}{\tan r_1\tan r_2}=\sin^2\tfrac{1}{2}a.$$

Now the angle of intersection of two curves is equal to the angle of intersection of their inverses. Hence the theorem is obvious.

Remark.—If the circles do not intersect in real points they may still be regarded as intersecting in two imaginary points, and at an imaginary angle. The student who is desirous of establishing the theorem independently for non-intersecting circles may apply the method used by Professor Casey (*Sequel to Euclid*) to establish the analogue *in plano*, viz.,

$$\frac{t^2}{r_1 r_2} = \frac{t'^2}{r_1' r_2'} \left(= 4 \sin^2 \tfrac{1}{2} a\right).$$

Cor. 1.—If r_1, r_2, r_3, r_4 be the spherical radii of four circles, t_{12}, t_{23}, &c., their common tangents, we have the relation

$$\sin \tfrac{1}{2} t_{23} \sin \tfrac{1}{2} t_{14} + \sin \tfrac{1}{2} t_{31} \sin \tfrac{1}{2} t_{24} + \sin \tfrac{1}{2} t_{12} \sin \tfrac{1}{2} t_{34} = 0.$$

[For if four circles of angular radii, ρ_1, ρ_2, ρ_3, ρ_4, touch a great circle in points A_1, A_2, A_3, A_4, we have

$$\sin \tfrac{1}{2} A_2 A_3 \sin \tfrac{1}{2} A_1 A_4 + \sin \tfrac{1}{2} A_3 A_1 \sin \tfrac{1}{2} A_2 A_4 + \sin \tfrac{1}{2} A_1 A_2 \sin \tfrac{1}{2} A_3 A_4 = 0.$$

Dividing this equation by $\sqrt{\tan \rho_1 \tan \rho_2 \tan \rho_3 \tan \rho_4}$, and inverting, it becomes, by the above theorem, the relation in question divided by $\sqrt{\tan r_1 \tan r_2 \tan r_3 \tan r_4}$; &c.]

This relation may be regarded as the criterion that four circles should be touched by a fifth. A direct deduction of it is given in Chap. XV., Prop. I.

When three of the circles reduce to points, this corollary gives us the condition of Art. 160, *Cor.* 4, that the circum-circle of a triangle should touch a given circle; and when the four circles become points, it reduces to the well-known analogue of Ptolemy's Theorem (Art. 25, Ex. 6).

Cor. 2.—Using the criterion of *Cor.* 1, it follows at once that the inscribed and three escribed circles of a spherical triangle are all touched by a fifth circle, viz., Hart's Circle.

[For the common tangents are $b + c$, $b - c$, $c + a$, $c - a$, $a + b$, $a - b$, the transverse common tangents to the in-circle with the ex-circles being taken because the contact of the circle with the in-circle differs from that with the ex-circles.]

Cor. 3.—If four circles be touched by a fifth, their angles of intersection are connected by the relation

$$\sin \tfrac{1}{2} a_{23} \sin \tfrac{1}{2} a_{14} + \sin \tfrac{1}{2} a_{31} \sin \tfrac{1}{2} a_{24} + \sin \tfrac{1}{2} a_{12} \sin \tfrac{1}{2} a_{34} = 0.$$

Examples.

1. The points where two circles are intersected by any orthogonal circle connect concurrently with the points where the great circle joining their poles meets them.

[For the two circles can be inverted into themselves, and the great circle joining their poles inverts into a circle cutting them orthogonally; hence the former set of points may be regarded as the inverses of the latter, and they, therefore, connect concurrently with them, the point of concurrence being the centre of inversion, *i.e.* a point on the radical axis of the circles.]

2. Every point of intersection (or point of contact) of a curve with the circle of inversion is also a point of intersection (or contact) of the inverse curve with the circle of inversion, and the angles of intersection of each curve with the circle are equal.

[This is a particular case of Art. 168, since the circle of inversion is its own inverse.]

3. Any circle passing through a pair of inverse points meets any curve and its inverse in corresponding points; and any such circle which touches

the curve also touches its inverse, or, more generally, intersects the curve and its inverse at equal angles at each pair of corresponding points.

[A circle passing through a pair of inverse points is its own inverse. Therefore, &c., by Art. 168.]

4. A system of coaxal circles of either species inverts into a system of coaxal circles of the same species; and if the centre of inversion be one of the common points of the system, the common points of the inverse system are the point diametrically opposite to the origin, and the inverse of the other common point.

5. Any system of concurrent great circles inverts into a system of coaxal circles; the common points of the system being the inverses of the common points of the great circles.

6. A system of circles having the same spherical centre (*i.e.* a copolar system) inverts into a system of circles cut orthogonally by a coaxal system, *i.e.* the system inverts into a coaxal system.

[For the copolar system of circles is cut orthogonally by their common system of secondaries, which inverts (Ex. 5) into a coaxal system, intersecting orthogonally the inverses of the copolar system.]

7. The great circle joining the origin to the common pole of the copolar system (Ex. 6) is the locus of the poles of the coaxal system into which the copolar system inverts, and the limiting points of the inverse system are the inverses of the two poles of the copolar system.

8. Any circle may be regarded as the inverse of any other with respect to two points as origin, viz. their two centres of similitude.

9. If the common points of a coaxal system of circles coincide, the system inverts from that common point as origin into a system of coaxal circles, whose common points coincide with the point diametrically opposite to the origin.

10. A system of circles having a common orthogonal circle can be inverted into a system of circles whose poles lie on a great circle.

[For the common orthogonal circle can be inverted into a great circle, which will intersect the inverses of the other circles orthogonally, and therefore pass through their poles.]

11. Hence, the locus of a point from which, as centre of inversion, a system of circles having a common orthogonal circle can be inverted into a system having their poles concyclic, is a circle.

NOTE.—*In plano* this circle coincides with the common orthogonal circle.

12. If three circles touch a circle passing through the point diametrically opposite to their radical centre, they also touch a circle passing through their radical centre.

[Invert from the radical centre as origin; then the three circles can be inverted into themselves, and the circle touching them inverts into a circle touching them and passing through the origin.]

13. Since a system of copolar circles is intersected orthogonally by their common system of secondaries, they are a coaxal system of the limiting point species, their common poles being their limiting points. So also a system of concurrent great circles forms a coaxal system of the common point species.

Hence a system of coaxal circles can be inverted into a copolar system by inverting the limiting points into two diametrically opposite points, and into a system of great circles by inverting the common points into two diametrically opposite points.

14. Hence show that if a variable circle intersects two fixed circles at constant angles, it intersects any circle coaxal with them at a constant angle, and touches two of the system.

[Invert the coaxal system into a system of concurrent great circles. Then the variable circle inverts into a circle cutting two of the great circles at constant angles, and it therefore intersects any other of them at a constant angle, and touches two of them.]

15. If a be the radius of inversion, every spherical figure inverts into its diametrically opposite when

$$\tan^2 \tfrac{1}{2} a = -1;$$

and every figure inverts into its reflection, with respect to the great circle of which the origin is the pole, when

$$\tan^2 \tfrac{1}{2} a = +1.$$

CHAPTER XIII.

PROJECTION.

SECTION I.

Spherical Projection.

175. We have already deduced many theorems concerning plane figures from the more general theorems concerning the corresponding spherical figures. We shall now proceed to show how, when we are given a theorem concerning a plane figure, we may deduce the analogous theorem for the corresponding spherical figure, a process which will recommend itself to the student, as it is a method of discovery, and enables us to rise from the particular to the more general.

176. **Surface and Centre of Projection.**—If all the points of any figure or curve be joined by right lines to a fixed point (O) in space, the joining lines will form a cone, having its vertex at O, and the section of this cone, by any surface, will constitute a figure called the *Projection* of the original figure on that surface.

Thus, for example, in the figure of Art. 11, a cone is formed by drawing lines from O to the contour of the figure $ABCD$. This cone meets a sphere, having its centre at O, in the figure $A'B'C'D'$, which is the projection of $ABCD$ on the sphere.

The surface on which any figure is projected is called the *Surface of Projection*, and the point O, from which the right lines are drawn, is called the *Centre of Projection.* In the example just cited the sphere is the surface of projection, and its centre (O) the centre of projection.

In what follows we shall deal only with the projection of figures from a plane upon a sphere, or *vice versa*, and in all cases either figure may be regarded as the projection of the other. In what is called *Spherical Projection,* the surface of projection is a sphere whose centre is the centre of projection.

177. From the very manner in which projection is performed, the following principles will appear self-evident:—

(α) Each point of the original figure will give rise to a point in the projected figure. Two such points are called *Corresponding Points.*

(β) Any two points of the original figure, and their two corresponding points in its projection, subtend the same angle at the centre of projection. Hence, by considering two consecutive points, it follows that—

(1) A tangent to the original figure is projected into a tangent to its projection at the corresponding point.

(2) If two curves touch each other at any number of points, their projections will touch each other at the corresponding points.

[In this case the cones from the centre of projection to the two curves touch each other along the edges to the points of contact of the curves.]

(γ) The surface of projection being a sphere, and the centre of projection its centre, any right line will be projected into a great circle.

[For the lines joining the centre of the sphere to the line will form a plane which will meet the sphere in a great circle, since it passes through the centre.]

The following Examples will serve to illustrate the method. In many cases simplicity and advantage will be gained by a judicious selection of the position of the centre of projection :—

1. The bisectors of the sides of a plane triangle are concurrent.	1. The bisectors of the sides of a spherical triangle are concurrent.

Take any sphere passing through the vertices of the plane triangle, the centre of projection being at its centre. Then the lines joining the centre of the sphere to the middle points of the sides of the plane triangle will pass through the middle points of the sides of the spherical triangle into which the plane triangle projects ; and therefore the bisectors of the sides of one project into the bisectors of the sides of the other. Hence, if one triad be concurrent, the planes joining them to the centre of the sphere will each pass through the line joining the point of concurrence to the centre of the sphere ; and the second triad must necessarily pass through the point where this line meets the triangle to which they belong.

2. The perpendiculars of a plane triangle are concurrent.	2. The perpendiculars of a spherical triangle are concurrent.

For the sake of clearness, take the plane of the triangle horizontal, and the centre of projection anywhere on the vertical through its orthocentre. Then the triad of planes joining the centre of projection to the perpendiculars of the triangle will be vertical, and perpendicular to the plane of the triangle, and to the corresponding sides of it, and therefore *to the planes joining the centre of projection to these sides.* But this latter triad meets a sphere (described round the centre of projection) in a spherical triangle; and, by what has been said, the former triad meets it in great circles passing through the vertices of this triangle, and perpendicular to its side : each of these perpendiculars must obviously pass through the point where the line joining the orthocentre to the centre of the sphere meets it.

3. The perpendiculars to the sides of a spherical triangle at their middle points are concurrent.

[Take the centre of projection on the perpendicular to the plane of the plane triangle at the centre of its circumcircle, and the circumcircle of one triangle will be the projection of the circumcircle of the other.]

4. The bisectors of the angles of a spherical triangle are concurrent.

[Take the centre of projection on the line perpendicular to the plane triangle at its in-centre.]

5. If a spherical hexagon be inscribed in a small circle, the intersections of the opposite sides lie on a great circle.

[Take the centre of projection on the perpendicular to the plane of the hexagon at the centre of the circle.]

6. If a spherical hexagon be circumscribed to a small circle, the arcs joining the opposite vertices are concurrent.

7. The arcs joining the vertices of a spherical triangle to the points of contact of its in-circle with the opposite sides are concurrent.

[Take the centre of projection on the perpendicular to the plane triangle at its in-centre.]

8. If the sides of a spherical triangle pass each through one of three fixed points lying on a great circle, and if two of the vertices move on two great circles, the locus of the third is a great circle passing through their intersection; and, reciprocally,

9. If a spherical quadrilateral be divided into any two others by an arc of a great circle drawn across it, the arc joining the intersections of the diagonals of the quadrilaterals into which it is divided passes through the intersection of the diagonals of the whole quadrilateral.

178. The preceding examples, since they deal only with the relative positions of points and lines, belong to the class termed *Graphical* by Poncelet. The following, since they deal with the relative magnitudes of lines, have been styled *Metrical*.

1. If from the angles A, B, C of a plane triangle lines be drawn meeting in a point P, and cutting the opposite sides in A', B', C'; then

$$\frac{BC' \,.\, CA' \,.\, AB'}{B'C \,.\, C'A \,.\, A'B} = 1.$$

1. If from the angles A, B, C of a spherical triangle great circles be drawn meeting in a point P, and cutting the opposite sides in A', B', C'; then

$$\frac{\sin BC' \sin CA' \sin AB'}{\sin B'C \sin C'A \sin A'B} = 1.$$

The plane triangle and its triad of concurrent lines will project into a spherical triangle, and a triad of concurrent arcs through its vertices. If, for brevity, we denote the segments of the sides of the spherical triangle by a_1, a_2, ; b_1, b_2; c_1, c_2; then O being the centre of projection, we have from the system of lines joining it to the plane triangle,

$$\frac{AC'}{BC'} = \frac{AO \sin AOC'}{BO \sin BOC'} = \frac{AO \sin c_1}{BO \sin c_2}.$$

Similarly,

$$\frac{BA'}{CA'} = \frac{BO \sin a_1}{CO \sin a_2}, \text{ and } \frac{CB'}{AB'} = \frac{CO \sin b_1}{AO \sin b_2};$$

therefore, etc. The same process will establish the following general theorem.

If the product of one set of right lines bears a constant ratio to the product of another set, each line of one set being in direction with a corresponding line of the other set, and the entire system of extremities of the lines of one set being the same as the entire system of the extremities of the other; then the continued product of the sines of the arcs into which one set is projected will bear the same ratio to the continued product of the sines of the arcs into which the other set is projected. (Cf. Art. 137.)

2. If a great circle be drawn across the sides of a spherical triangle, the product of the sines of one set of alternate segments is equal to the product of the sines of the other set.

3. If from any point P on a sphere a great circle be drawn cutting any number of fixed great circles in points A_1, A_2, A_3, &c., and if a point X be taken on it, such that

$$\cot PX = \Sigma \cot PA,$$

the locus of X is a great circle.

4. If a line be drawn parallel to a pair of opposite sides of a parallelogram $ABCD$, meeting AD in M, and BC in N; and if another line parallel to AD and BC meet AB in P, and CD in Q; then, obviously,

$$AP \,.\, BN \,.\, CQ \,.\, DM = PB \,.\, NC \,.\, QD \,.\, MA.$$

Hence, if from the extremities of the third diagonal of a complete quadrilateral on a sphere arcs be drawn, each cutting a pair of opposite sides, the products of the sines of the alternate segments are equal.

SECTION II.

Stereographic Projection.

179. **Definition.**—The *Stereographic Projection* of a spherical curve is the figure obtained by projecting the curve on the plane of a great circle, the pole of this great circle being the centre of projection. Thus, the centre of projection is a point on the surface of the sphere, and the surface of projection is a plane passing through the centre of the sphere, parallel to the tangent plane at the centre of projection.

180. **Theorem.**—*The stereographic projection of any circle, great or small, not passing through the centre of projection, is a circle.*

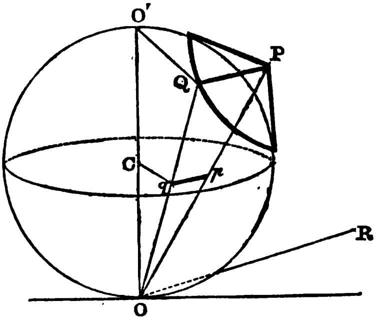

Fig. 69.

Let Q (fig. 69) be any point on the circle, and let P be the vertex of a cone touching the sphere around the circle. Then, if O be the centre of projection, and p and q the projections of P and Q, the plane OPQ will meet the plane of projection in a line pq, the tangent plane at O in a parallel line OR, and the sphere in a circle to which OR and PQ are tangents, OQ being their chord of contact. Hence the angles PQO, ROQ, and Qqp are equal, and therefore, since the triangles OQP and Oqp have one angle common, and the angles at Q and q supplemental, we have

$$pq = PQ\,\frac{Op}{OP} = \text{constant}.$$

Hence q describes a circle round p as centre.

Cor.—Any circle passing through the centre of projection is projected into a right line, viz. the line in which the plane of the circle meets the plane of projection.

181. Lemma.—*Two right lines containing an angle will be projected into two others containing an equal angle, when the line joining the centre of projection to the vertex of the given angle makes equal angles, in opposite directions, with the plane of projection and the plane of the given angle, and when it is also perpendicular to the line of intersection of these planes.*

Let the line joining C, the centre of projection to V, the vertex of the given angle, meet the plane of projection, $MA'B'N$ (fig. 70), in the point O. With O as centre, and a radius OC, describe a sphere. Through O draw OA and OB parallel to the lines containing the given angle, and let OA' and OB' be their projections. Draw a great circle, $MPQN$, through MN, having C for pole. Then, by the conditions of the problem, since the planes ABN and $A'B'N$ are equally inclined to OV, they are equally inclined to PQN; and since the angles at P and Q are right, we have, from the triangles ANP and $A'NP$, $AP = A'P$, and therefore $AN = A'N$. Similarly, $BN = B'N$. Hence $AB = A'B'$; therefore, &c.

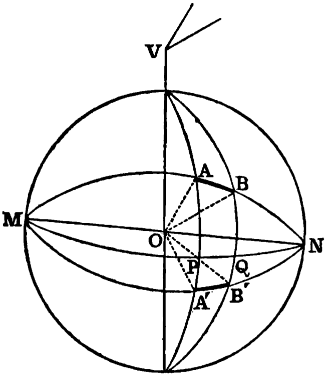

Fig. 70.

182. **Angles unaltered by Stereographic Projection.**—*The angles of intersection of any two spherical curves are equal to the angles of intersection of their projections at the corresponding point.*

Let Q be a point of intersection of two curves (fig. 69). Draw tangent lines to the curves at Q. The angle between the tangents is equal to the angle of intersection of the curves. Now, the plane of the tangents at Q being a tangent plane to the sphere, meets the tangent plane at O, and therefore the plane of projection, in a line perpendicular to OQ, since the line of intersection of two tangent planes to a sphere is perpendicular to their chord of contact, and it is clear that OQ meets the tangent plane at Q, or the plane of the angle, and the plane of projection at equal angles in opposite directions. Hence the theorem follows by the foregoing lemma.

183. **Theorem.**—*The stereographic projection of any spherical curve is its inverse with respect to the centre of projection as origin.*

Let Q (fig. 69) be any point on the curve. Join Q to O', the point diametrically opposite to O, and let the projection of Q be joined to C, the centre of the sphere. Then the angle OCq is right, and OQO' is also right, since it is in a semicircle. Therefore the quadrilateral $CO'Qq$ is inscribable in a circle, and

$$OQ \,.\, Oq = OO' \,.\, OC = 2r^2.$$

Hence q is the inverse of Q with respect to the centre of projection as origin, and a radius of inversion equal to $r\sqrt{2}$, where r is the radius of the sphere.

184. **Stereographic Projection of a Spherical Triangle.**

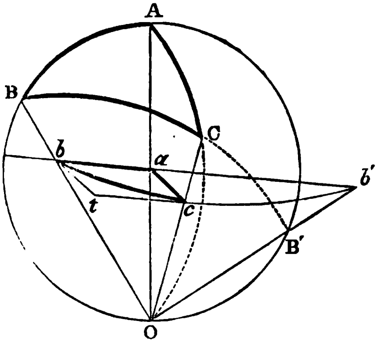

Fig. 71.

Let ABC (fig. 71) be a spherical triangle, and O the point diametrically opposite to A, the centre of projection.

According to what has been shown, the sides AB and AC project into the right lines ab and ac, and the side BCB' projects into a circle bcb'. Again, if tangents bt and ct be drawn to this circle at b and c, the angles bac, tba, and tca are equal to A, B, and C, respectively (Art. 182). Therefore, if the chords bc and $b'c$ be drawn, we have

$$A + B + C - \pi = 2tbc = 2bb'c \text{ (\textit{Euc.} III., xxxii.).}$$

Hence the angle $bb'c$ is half the spherical excess of the triangle ABC.

Cor. 1.—Having given the base AB and the area of the spherical triangle, the locus of the vertex C is a small circle passing through the points diametrically opposite to the extremities of the base.

[For, since the area is given, the angle $bb'c$ is given, and the locus of c is a fixed line $b'c$, and therefore the locus of C is the circle in which the plane Ocb' meets the sphere.]

Cor. 2.—The angles of the triangle abc are

$$A,\ B - \tfrac{1}{2}E,\ C - \tfrac{1}{2}E,\ \textit{i.e.}\ A,\ \tfrac{1}{2}E_2,\ \tfrac{1}{2}E_3.$$

185. *To express the sides of the triangles abc and $ab'c$ in terms of the sides of ABC.*

Let the radius of the sphere be of unit length. Then

$$ab = \tan AOB = \tan \tfrac{1}{2} c. \qquad (1)$$

and

$$ac = \tan AOC = \tan \tfrac{1}{2} b; \qquad (2)$$

also, since the triangles BOC and bOc are similar, we have

$$bc = BC \cdot \frac{Ob}{OC} = BC\frac{Ob \cdot OB}{OB \cdot OC} = \frac{2BC}{OB \cdot OC},$$

since $Ob \cdot OB = 2r^2 = 2$ (see Art. 183).

Therefore

$$bc = \frac{\sin \frac{1}{2} a}{\cos \frac{1}{2} b \cos \frac{1}{2} c}. \qquad (3)$$

Now, let ac meet the circle bcb' again in c'; then c' is the projection of C', the point diametrically opposite to C; and since the angle BOB' is right, we have

$$ab' = \tan AOB' = \cot \tfrac{1}{2} c, \qquad (4)$$

and $$ac' = \tan AOC' = \cot \tfrac{1}{2} b; \qquad (5)$$

therefore $$bb' = ab + ab' = 2 \operatorname{cosec} c, \qquad (6)$$

and $$cc' = ac + ac' = 2 \operatorname{cosec} b; \qquad (7)$$

also $$b'c = \frac{2B'C}{OB' . OC} = \frac{\cos \frac{1}{2} a}{\cos \frac{1}{2} b \sin \frac{1}{2} c}. \qquad (8)$$

It is clear that $ab'c$ is the projection of the colunar triangle $AB'C$, and that abc' is the projection of ABC'. Hence the equations (4), (5), (8), may be written down at once from the equations (1), (2), (3). Or thus:—Since $Oa = 1$, we have $ab . ab' = 1$, and $ac . ac' = 1$, so also $bb' = Ob . Ob'$, and $cc' = Oc . Oc'$, &c.

186. **Area of a Spherical Quadrilateral.**—Let $ABCD$ be the quadrilateral; a, b, c, d, its sides AB, BC, CD, DA, respectively; δ and δ' its diagonals AC and BD.

Project the quadrilateral stereographically, taking the point diametrically opposite to A for centre of projection. The projection will have two sides right lines, and two arcs of circles; and if we draw tangents bt and ct to the arc bc, and tangents ct' and dt' to the arc cd, we have

$$E = A + B + C + D - 2\pi = 2tcb + 2t'cd = 2bc'd,$$

c and *c'* being the points of intersection of the circles into which *BC* and *CD* project. Therefore the triangle *bc'd* gives

$$\sin^2 \tfrac{1}{4} E = \sin^2 \tfrac{1}{2}\, b\hat{c'}d = \frac{(bd + bc' - dc')(bd + dc' - bc')}{4bc' \,.\, dc'}.$$

But

$$bd = \frac{\sin \frac{1}{2}\delta'}{\cos \frac{1}{2}a \cos \frac{1}{2}d},\quad bc' = \frac{\cos \frac{1}{2} b}{\cos \frac{1}{2}a \sin \frac{1}{2}\delta},\quad dc' = \frac{\cos \frac{1}{2}c}{\cos \frac{1}{2}d \sin \frac{1}{2}\delta}.$$

Therefore

$$\sin^2 \tfrac{1}{4} E = \frac{(\sin \frac{1}{2}\delta \sin \frac{1}{2}\delta' + \cos \frac{1}{2}a \cos \frac{1}{2}c - \cos \frac{1}{2}b \cos \frac{1}{2}d)(\sin \frac{1}{2}\delta \sin \frac{1}{2}\delta' - \cos \frac{1}{2}a \cos \frac{1}{2}c + \cos \frac{1}{2}b \cos \frac{1}{2}d)}{4 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c \cos \frac{1}{2}d}$$

(See Art. 112.)

Examples.

1. The inscribed and escribed circles of a spherical triangle are all touched by another circle, viz. Dr. Hart's. Hence, by stereographic projection, we find *Dr. Hart's extension of Feuerbach's Theorem*, viz., "If the three sides of a plane triangle be replaced by three circles, then the four circles touching these, which correspond to the in-circle and ex-circles of a plane triangle, are all touched by another circle."

2. Establish the formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

[The triangle *abc* gives

$$(bc)^2 = (ca)^2 + (ab)^2 - 2\,(ca)\,(ab) \cos A.$$

Substituting from Art. 185, the required result is obtained.]

3. Prove that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

[If *d* be the diameter of the circle *bcb'*, the angle which *bb'* subtends at any point of its circumference being *tbb'*, or *B*, we have

$$bb' = d \sin B, \text{ and } cc' = d \sin C.$$

Therefore

$$\frac{\sin B}{\sin C} = \frac{bb'}{cc'} = \frac{\sin b}{\sin c}. \qquad \text{(Art. 185.)]}$$

4. Deduce the analogies of Napier and Delambre.

[The triangle *abc* gives

$$\frac{\tan \frac{1}{2}(\hat{b} - \hat{c})}{\cot \frac{1}{2}\hat{a}} = \frac{ac - ab}{ac + ab},$$

which transforms, by Arts. 184 and 185, into

$$\tan \tfrac{1}{2}(B - C) = \frac{\sin \frac{1}{2}(b - c)}{\sin \frac{1}{2}(b + c)} \cot \tfrac{1}{2} A.$$

Again, in any plane triangle,

$$a = \frac{(b + c) \sin A}{\sin B + \sin C} = \frac{(b - c) \sin A}{\sin B - \sin C};$$

or,

$$a = \frac{(b + c) \sin \frac{1}{2} A}{\cos \frac{1}{2}(B - C)} = \frac{(b - c) \cos \frac{1}{2} A}{\sin \frac{1}{2}(B - C)};$$

Applying this formula to the triangle *abc* (fig. 71), we find

$$\sin \tfrac{1}{2} a = \frac{\sin \frac{1}{2}(b + c) \sin \frac{1}{2} A}{\cos \frac{1}{2}(B - C)} = \frac{\sin \frac{1}{2}(b - c) \cos \frac{1}{2} A}{\sin \frac{1}{2}(B - C)}.\Big]$$

5. Establish Cagnoli's formula for $\sin \frac{1}{2} E$.

[We have, from the triangle *ab'c*,

$$\frac{\sin \hat{b}'}{\sin \hat{a}} = \frac{ac}{b'c};$$

or,

$$\sin \tfrac{1}{2} E = \sin \hat{a} \, \frac{ac}{b'c} = \frac{\sin \frac{1}{2} b \sin \frac{1}{2} c}{\cos \frac{1}{2} a} \sin A; \ \&\text{c.}]$$

6. Show that

$$\cos \tfrac{1}{2} E = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c},$$

by applying to the triangle *ab'c* the formula

$$ab' = ac \cos \hat{a} + cb' \cos \hat{b}'.$$

7. Prove that

$$\tan \tfrac{1}{4} E = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)},$$

by applying the ordinary expression for the tangent of half an angle of a triangle to the angle *ab'c*.

8. If R be the circumradius of a spherical triangle, show that

$$\tan R = \frac{2 \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin \tfrac{1}{2} c}{n}.$$

[Take the centre of projection diametrically opposite to A, then the spherical diameter of the circumcircle which passes through A projects into a line, viz., a diameter of the projection of the circumcircle, and the latter diameter is obviously equal to $\tan R$. But in a plane triangle we have

$$R = \frac{abc}{4\Delta}, \text{ \&c.]}$$

9. Having given two sides of a spherical triangle, determine when its area is a maximum.

[Let AB and AC be given. Suppose AB fixed, then B' is fixed, and in the triangle $ab'c$ we have ab' and ac given; therefore the angle $ab'c$ is a maximum when acb' is a right angle, that is, when

$$ab'c + cab' = 90^\circ;$$

or when

$$A = B + C.$$

Again, since $ab' = \cot \tfrac{1}{2} c$ and $ac = \tan \tfrac{1}{2} b$,

the cases presented when $b + c <=> \pi$ follow, as in Art. 113.]

10. Any system of coaxal, or copolar circles, on the sphere, projects stereographically into a system of coaxal circles.

[For either system has a system of orthogonal circles.]

11. Four circles on a sphere touch a fifth circle; prove that their common tangents are connected by the relation

$$\sin \tfrac{1}{2} t_{23} \sin \tfrac{1}{2} t_{14} + \sin \tfrac{1}{2} t_{31} \sin \tfrac{1}{2} t_{24} + \sin \tfrac{1}{2} t_{12} \sin \tfrac{1}{2} t_{34} = 0.$$

[The four circles project into four circles, touched by a fifth in the plane of projection, and therefore the common tangents of the projected circles are connected by the relation

$$t_{23} t_{14} + t_{31} t_{24} + t_{12} t_{34} = 0.$$

But since the angle (α) of intersection of any two curves is unaltered by the projection, we have

$$\frac{\sin^2 \frac{1}{2} t_{12}}{\tan r_1 \tan r_2} = \sin^2 \tfrac{1}{2} \alpha = \left(\frac{t_{12}}{4 r_1 r_2}\right)^2; \text{ therefore, \&c.]}$$

The above Examples might be multiplied to almost any extent. They are, however, sufficient in number and variety to illustrate the working and power of the method of Stereographic Projection.

CHAPTER XIV.

SPHERICAL RECIPROCATION.

187. **Polar Reciprocal Figures. Definitions.**—Being given any spherical curve F, and a fixed small circle, we can generate another curve F' by taking the poles of the tangents to F with respect to the circle. Thus F' is the locus of the poles of the tangents to F, and is, for this reason, termed the *Polar Curve* of F. Since to every point of F' there will *correspond* a tangent great circle to F, it is clear (Art. 152) that *the great circle joining any two points of the former will be the polar of the intersection of the corresponding tangents to the latter.* Now, let us suppose that the tangents to F are infinitely near; then, the corresponding points on F' will be infinitely near, and the great circle joining them will be a tangent to F'. But the intersection of the consecutive tangents to F is a point on the curve; therefore—

If a tangent to F at a point A correspond to a point A' on F'; then the tangent to F' at A' will correspond to the point A on F.

Hence the relation between the curves is reciprocal, and they are therefore said to be *Polar Reciprocals*, each of the other, with respect to the given circle. The spherical centre of the circle, its angular radius, and the circle itself, are termed respectively the *Centre*, *Radius*, and *Circle of Reciprocation.*

Polar reciprocal figures possess the following evident properties :—

(1) To every point related to one corresponds a great circle related to the other.

(2) If any number of points of one lie on a given great circle, the corresponding great circles of the other pass through a point, viz. the pole of the given great circle, and *vice versa.* More generally, if any point connected with F describes a curve f, the corresponding great circle of F' envelopes a curve f' the polar reciprocal of f, and *vice versa.*

(3) If two curves have a common point, their polar reciprocals have a common tangent, viz. the polar of the point.

Thus, the intersections of a pair of curves correspond to the common tangents of their polar reciprocals; and the intersections of the common tangents of one pair correspond to chords of intersection of the other pair.

(4) Hence, if two curves touch, their polar reciprocals also touch.

[For the first pair have a point common, and also the tangent at that point common; therefore the second pair have a tangent common, and also the point of contact common].

(5). The number of points in which one is intersected by any great circle is equal to the number of tangents which can be drawn from any point to the other.

(6) Any point and its polar, with respect to one, will

correspond to a great circle, and its pole, with respect to the other.

(7) The anharmonic ratio of any spherical row connected with one figure is equal to the anharmonic ratio of the corresponding pencil connected with the other.

(8) The polar reciprocal of a polygon, inscribed to a small circle *copolar* with the circle of reciprocation, is a polygon escribed to a small circle also copolar with the circle of reciprocation; the points of contact with the circle in the latter corresponding to the vertices of the polygon in the former.

188. **Duality of Graphical Relations or Theorems of Position.**—From the above properties of polar reciporcal curves it will appear that every theorem of position (*i. e.* one referring only to the relative positions of points and great circles, and not involving magnitudes of angles or angular distances) is twofold, and from it we can derive another by interchanging the words "points" and "great circles." This principle is illustrated by the following examples:—

1. The points of intersection of the opposite sides of a hexagon in a small circle lie on a great circle.	1. The great circles joining the opposite vertices of a hexagon escribed to a circle pass through a point.
2. If two of the vertices of a triangle move on fixed great circles, while the sides pass through three fixed concyclic points, the locus of the third vertex is a great circle.	2. If two sides of a triangle pass through fixed points, while its vertices move on three fixed concurrent greatcircles, the envelope ofthe third side is a point.

189. **Metrical Relations.**—In order to transform

theorems involving metrical properties (*i. e.* magnitudes of angles and distances) it is necessary to connect—

(1) The distance from the origin of any point of one figure with that of the corresponding great circle of the other.

(2) The angles of one figure with the parts of the other.

The first of these relations is obtained from the fundamental equation (fig. 57)—

$$\tan PO \tan PO' = \tan^2 a.$$

In plane geometry the second relation is obtained at once, since the angle between any two lines is equal to, or is the supplement of, the angle which the line joining their poles subtends at the centre of reciprocation. On the sphere, however, the method becomes defective, since the angle between any two great circles of one figure has no direct representative on the reciprocal figure. However, as yet we have in no way limited the circle of reciprocation, and we may therefore anticipate that by restricting it to obey certain chosen conditions, we may be able to remove the foregoing defect, and render the method of reciprocal polars applicable to metrical as well as to graphical theorems.

That this is so will be seen from the following Article:—

190. **Case when** $\tan^2 a = -1$. **Supplementary Figures.**—Since the polar reciprocal of a great circle is a point, it is possible by a proper choice of the circle of reciprocation to reciprocate a great circle into its spherical

centre. Thus, if the radius of reciprocation (a) be given by the equation $\tan^2 a = -1$; then (fig. 57), for any point O and its polar, we have

$$\tan PO \tan PO' = -1.$$

Hence $$OO' = 90^\circ.$$

Therefore *any great circle reciprocates into its spherical centre when the radius of reciprocation is imaginary, and given by the equation* $\tan a = \sqrt{-1}$.

Thus, the spherical centre of any great circle is its pole with respect to an imaginary small circle, the tangent of whose radius is $\sqrt{-1}$, and the spherical centre of which may have any position on the sphere.

The circle of reciprocation being so chosen, the following evident properties enable us to deal with *metrical theorems* :—

(1) *The angular distance of any point of a curve from the origin is the complement of the angular distance from it of the corresponding great circle of the reciprocal curve, the distances being measured in opposite directions from the origin.*

(2) *The angle between two great circles is equal to, or the supplement of, the angular distance between their poles.*

191. It is now clear that, with respect to such a small circle, the polar reciprocal of any figure or theorem coincides with what we have heretofore called the supplemental figure, or theorem; the latter name having been applied because the arcs and angles of one figure are the supplements of the angles and arcs of the other. We may therefore write down the following corollaries :—

Cor. 1.—The reciprocal of a small circle is a copolar small circle of complementary angular radius.

Cor. 2.—The reciprocal of any triangle or polygon is its supplementary triangle or polygon, the angles of one polygon being the supplements of the sides of the other.

Cor. 3.—Hence, if a polygon be inscribed (or escribed) to a small circle, its polar polygon will be escribed (or inscribed) to a small circle of complementary radius.

Cor. 4.—The spherical excess of any polygon (or curve), added to the perimeter of its polar polygon (or curve), is constant, and equal to 2π.

[The excess of the polygon (Art. 111) is $\Sigma A - (n-2)\pi$, and the perimeter of its polar polygon is $n\pi - \Sigma A$; therefore, &c.]

192. We shall conclude this Chapter with a list of reciprocal properties and theorems, the reciprocation being with respect to the imaginary circle of Art. 190, in which case, as has been already remarked, the reciprocal figure coincides with what we have heretofore known as the *Supplemental Figure.* We group, by way of illustration of the method, in pairs, as reciprocal theorems, some of the Exercises which have already appeared in the preceding pages as apparently independent of each other.

Let the primitive figure, F, consist of a triangle, with its associated points, arcs, and circles; then the polar figure will consist of a triangle with its associated arcs, points, and circles; and the relative positions of the great circles of either figure, and of their poles in the other, are shown in the subjoined list:—

PRIMITIVE FIGURE.	RECIPROCAL FIGURE.
1. A great circle passing through the vertex.	1. A point on the base.
2. The segments of the vertical angle.	2. The segments of the base.
3. Two great circles, at a given angle, passing through the vertex.	3. Two points on the base intercepting on it an equal angle.
4. Internal and external bisectors of the vertical angle.	4. Mid-point of base and a point on it 90° from mid-point, *i.e.* internal and external bisectors of the base.
5. Perpendiculars.	5. Points on the sides 90° distant from the opposite vertices.
6. Orthocentre.	6. Axis of perspective of the triangle and its polar triangle.
7. Medians.	7. Intersections of the bisectors of the angles with the opposite sides.
8. Three or more concurrent arcs.	8. Three or more concyclic points.
9. Points equidistant from the sides; *i.e.* the in-centre and three ex-centres.	9. Great circles equidistant from the vertices.
10. The arcs joining the vertices to the in-centre.	10. The angles made with the sides by the axis of perspective of the polar triangle and its mid-point side triangle.
11. The arcs joining the vertices to the ex-centres.	11. The angles made with the sides by the arcs joining their middle points.
12. The angles made by the medians with the sides.	12. The lengths of the bisectors of the angles.
13. A variable great circle cutting the base at a fixed angle.	13. A point describing a small circle round the vertex as pole.
14. Perpendiculars at the middle points of the sides.	14. Points on the bisectors of the angles 90° from the corresponding vertices.
15. The angles made by any transversal with the sides.	15. The angular distances of the pole of the circle from the vertices.
16. The angles between the bisectors of the angles, internal and external, at their point of concurrence.	16. The arcs joining the middle points of the sides of a triangle and its colunars.

193. From the preceding relations the following are manifest:—

1. Since the bisectors of the angles pass in triads through four points, *the internal and external bisectors of the sides lie three and three on four great circles.* [Cf. Arts. 122, 20.]

2. Since the four points in Ex. 1 are the poles of the in- and ex-circles, *the arcs joining the middle points of the sides are each equidistant from the vertices.*

3. Since the six bisectors of the sides (internal and external) pass in triads through four points, *the intersections of the bisectors of the angles with the opposite sides lie three and three on four great circles.*

4. Since the perpendiculars at the middle points of the sides pass in triads through the poles of the circumcircles of a triangle and its three colunars, *the bisectors of the angles meet the corresponding sides of the polar triangle in six points, which lie in triads on four great circles.*

5. From the theorem of Art. 40 may be deduced the value of the angle θ made with the base by any great circle passing through the vertex, in terms of the base angles and the segments, α and β, of the vertical angle C; thus,

$$\cos A \sin \alpha - \cos B \sin \beta = \pm \cos \theta \sin C.$$

[Cf. Art. 35, Ex. 5.]

6. From the values obtained in Art. 42, for the internal and external bisectors of the base, we find expressions for the angles made with the base by the internal and external bisectors of the vertical angle. See Art. 35, Exs. 9 and 10.

7. It follows from the preceding Article that the result in Art. 35, Ex. 6, gives at once Art. 42, Ex. 5; and conversely.

8. From the expressions for the bisectors of the vertical angle will follow the values of the angles made by the bisectors of the base with the base.

[Cf. Art. 39, Ex. 1, and Art. 39, Ex. 11.]

9. Since the perpendiculars of a triangle are concurrent, *three points on the sides, 90° distant from the opposite vertices, are concyclic.* And conversely; if two points, X and Y, be taken on the sides a and b of a triangle, 90° distant from the opposite vertices, the great circle XY will meet the base in a point 90° from the third vertex. Now, since a triangle ABC and the

three connectors, AX, BY, CZ, form a complete quadrilateral, of which the connectors are the diagonals, we obtain a theorem already proved in Art. 145, Ex. 2, and Art. 125, *Cor.*

If any two of the diagonals of a complete quadrilateral are quadrants, the third diagonal is also a quadrant.

10. The relation connecting the three pairs of opposite connectors of any complete quadrilateral with the angles contained by each pair involves reciprocally one connecting the three pairs of opposite angles with the diagonals.

[Cf. *Exam. Papers*, V., Ex. 1, and IX., Ex. 1.]

11. If O denote the in-centre of a triangle,

$$BOC + COA + AOB = 2\pi.$$

Hence—*The arcs joining the middle points of the sides of the three colunar triangles are together equal to two right angles.*

(Art. 121, Ex. 9.)

12. Having given the base of a triangle = 90°, and the sum of the squares of the cotangents of the base angles, find the locus of the vertex.

[The reciprocal problem is "to find the envelope of the base of a right-angled triangle when $\cot^2 a + \cot^2 b$ is given." See Art. 56, Ex. 6 (*b*); therefore, &c.]

13. "Having given the base c and $l \cot A + m \cot B$, find the locus of the vertex" (Chap. III., Misc. Ex. 42) involves the reciprocal problem; "having given the vertical angle C and $l \cot a + m \cot b$, find the envelope of the base." (Art. 39; Ex. 10.)

14. Having given the vertical angle C and $l \cos A + m \cos B$, find the envelope of the base. [Art. 42, Ex. (1).]

15. When the vertical angle of a triangle and perimeter are given, the base touches the ex-circle which, by the given conditions, is fixed in position; hence *when the base and area are given, the locus of the vertex is a small circle.* (See Art. 101.)

16. If a quadrilateral be circumscribed to a small circle, the product of the cosines of the halves of the internal angles of intersection of opposite sides is equal to the sum of the products of the cosines of the halves of the opposite angles of the quadrilateral. (See Art. 25, Ex. 6.)

CHAPTER XV.

MISCELLANEOUS PROPOSITIONS.

PROP. 1.—**To find a relation connecting the common tangents of four circles which touch a fifth.**

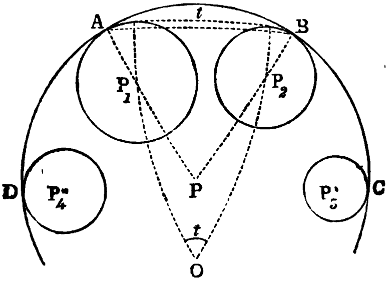

Fig. 72.

Let P_1, P_2, P_3, P_4 (fig. 72) be the poles of the four circles, r_1, r_2, r_3, r_4, their angular radii, P the pole, and R the angular radius of the circle touching them at the points A, B, C, D. Then (Art. 33) the triangle P_1PP_2 gives us, if $P_1P_2 = \delta_{12}$,

$$\sin^2 \tfrac{1}{2} P = \frac{\cos(r_1 - r_2) - \cos\delta_{12}}{2\sin(R - r_1)\sin(R - r_2)}.$$

Therefore

$$\sin \tfrac{1}{2} AB = \sin R \sqrt{\frac{\cos(r_1 - r_2) - \cos\delta_{12}}{2\sin(R - r_1)\sin(R - r_2)}}.$$

But if t_{12} denote the common tangent to the circles P_1 and P_2, we have, from the triangle P_1OP_2,

$$\sin \tfrac{1}{2} t_{12} = \sin \tfrac{1}{2} O = \sqrt{\frac{\cos(r_1 - r_2) - \cos\delta_{12}}{2\cos r_1 \cos r_2}}$$

Hence

$$\sin\tfrac{1}{2}AB = \sin\tfrac{1}{2}t_{12}\sin R\sqrt{\frac{\cos r_1\cos r_2}{\sin(R-r_1)\sin(R-r_2)}}$$

Now, since the quadrilateral $ABCD$ is cyclic, we have

$$\sin\tfrac{1}{2}BC\sin\tfrac{1}{2}AD + \sin\tfrac{1}{2}CA\sin\tfrac{1}{2}BD + \sin\tfrac{1}{2}AB\sin\tfrac{1}{2}CD = 0.$$

Therefore, by substitution, we find

$$\sin\tfrac{1}{2}t_{23}\sin\tfrac{1}{2}t_{14} + \sin\tfrac{1}{2}t_{31}\sin\tfrac{1}{2}t_{24} + \sin\tfrac{1}{2}t_{12}\sin\tfrac{1}{2}t_{34} = 0.$$

(See Art. 174, *Cor.* 1.)

Prop. 2.—**If σ_1, σ_2, σ_3 denote the symmedians of a triangle from the angles A, B, C, respectively, to prove that**

$$\tan\sigma_1 = \sqrt{\frac{\sin^2 b + \sin^2 c + 2\sin b\sin c\cos A}{\cot b\sin c + \cot c\sin b}},$$

with similar expressions for σ_2 and σ_3.

For let x and z be the segments of the angle A, made by the median β. Then we have

$$\cot b\sin x + \cot c\sin y = \cot\beta\sin A; \quad \text{(Art. 55, \textit{Cor.} 2.)}$$

also $$\cot c\sin x + \cot b\sin y = \cot\sigma\sin A,$$

and $$\sin c\sin x - \sin b\sin y = 0.$$

Eliminating x and z from these three equations, and substituting for $\sin\beta$ in the result the value given in Exam. Paper VII., Ex. 4, the above expression is obtained, since $2\cos\beta\cos\frac{1}{2}a = \cos b + \cos c$.

Prop. 3.—**Having given the three perpendiculars, α, β, γ, of a triangle; construct it.**

From the equations $\sin a\sin\alpha = \sin b\sin\beta = \sin c\sin\gamma = 2n$ (1), we have

$$(\operatorname{cosec}\alpha + \operatorname{cosec}\beta + \operatorname{cosec}\gamma)^2 - 1 = \left(\frac{\sin a + \sin b + \sin c}{2n}\right)^2 - 1$$

$$= \left[\frac{\sin s + \sin(s-a) + \sin(s-b) + \sin(s-c)}{2n}\right]^2.$$

[Art. 85 (1).]

Similarly,

$$(\operatorname{cosec}\beta + \operatorname{cosec}\gamma - \operatorname{cosec}\alpha)^2 - 1 = \left(\frac{\sin b + \sin c - \sin a}{2n}\right)^2 - 1$$

$$= \left[\frac{\sin s + \sin(s-a) - \sin(s-b) - \sin(s-c)}{2n}\right]^2; \qquad \text{[Art. 85 (2).]}$$

with similar expressions for $\operatorname{cosec}\gamma + \operatorname{cosec}\alpha - \operatorname{cosec}\beta$, &c.

Now let the left members of these four equations be denoted by $\kappa, \kappa_1, \kappa_2, \kappa_3$; and, by addition,

$$2\sin s = n(\kappa + \kappa_1 + \kappa_2 + \kappa_3);$$

also
$$2\sin(s-a) = n(\kappa + \kappa_1 - \kappa_2 - \kappa_3),$$

and
$$2\sin(s-b) = n(\kappa - \kappa_1 + \kappa_2 - \kappa_3),$$

$$2\sin(s-c) = n(\kappa - \kappa_1 - \kappa_2 + \kappa_3);$$

therefore, by multiplication and reduction,

$$4n = \sqrt{(\kappa + \kappa_1 + \kappa_2 + \kappa_3)(\,.\;.\;.\,)(\,.\;.\;.\,)(\,.\;.\;.\,)},$$

and is therefore known. Hence we may find the sides from (1).

Prop. 4.—**Having given the four sides of a quadrilateral inscribed in a circle; to determine the diagonals and angles.**

Let $\alpha, \beta, \gamma, \delta, S$ be the sides and area of the chordal quadrilateral; then

$$2S = (\alpha\beta + \gamma\delta)\sin\widehat{\alpha\beta} = (\beta\gamma + \alpha\delta)\sin\widehat{\beta\gamma}.$$

Hence

$$\frac{\beta\gamma + \alpha\delta}{\alpha\beta + \gamma\delta} = \frac{\sin\widehat{\alpha\beta}}{\sin\widehat{\beta\gamma}} = \text{ratio of diagonals.}$$

Therefore if a, b, c, d denote the sides; δ, δ' the diagonals of the spherical quadrilateral, we have

$$\frac{\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c + \sin\tfrac{1}{2}a\sin\tfrac{1}{2}d}{\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b + \sin\tfrac{1}{2}c\sin\tfrac{1}{2}d} = \frac{\sin\tfrac{1}{2}\delta}{\sin\tfrac{1}{2}\delta'}; \qquad (1)$$

also
$$\sin\tfrac{1}{2}a\sin\tfrac{1}{2}c + \sin\tfrac{1}{2}b\sin\tfrac{1}{2}d = \sin\tfrac{1}{2}\delta\sin\tfrac{1}{2}\delta', \qquad (2)$$

two equations which give at once

$$\sin^2 \tfrac{1}{2}\delta^*, \text{ and } \sin^2 \tfrac{1}{2}\delta'.$$

Otherwise thus:—Applying the formula of Art. 117 to the triangles on the base δ, it follows that

$$\frac{\sin^2 \frac{1}{2}\delta - \sin^2 \frac{1}{2}b - \sin^2 \frac{1}{2}c}{\sin \frac{1}{2}b \sin \frac{1}{2}c} + \frac{\sin^2 \frac{1}{2}\delta - \sin^2 \frac{1}{2}a - \sin^2 \frac{1}{2}d}{\sin \frac{1}{2}a \sin \frac{1}{2}d} = 0;$$

therefore, &c.

Again,

$$\sin^2 \tfrac{1}{2}\widehat{ab} = \frac{\sin^2 \frac{1}{2}\delta - \sin^2 \frac{1}{2}(a-b)}{\sin a \sin b}.$$

Substituting in this equation the value of $\sin^2 \frac{1}{2}\delta$, already obtained, and reducing, we find

$$\sin^2 \tfrac{1}{2}\widehat{ab} = \frac{\cos \frac{1}{4}(\overline{a-b} + \overline{c-d}) \cos \frac{1}{4}(\overline{a-b} - \overline{c-d}) \sin \frac{1}{2}(s-a) \sin \frac{1}{2}(s-b)}{\cos \frac{1}{2}a \cos \frac{1}{2}b (\sin \frac{1}{2}a \sin \frac{1}{2}b + \sin \frac{1}{2}c \sin \frac{1}{2}d)}. \tag{3}$$

Similarly,

$$\sin^2 \tfrac{1}{2}\widehat{cd} = \frac{\cos \frac{1}{4}(\overline{a-b} + \overline{c-d}) \cos \frac{1}{4}(\overline{a-b} - \overline{c-d}) \sin \frac{1}{2}(s-c) \sin \frac{1}{2}(s-d)}{\cos \frac{1}{2}c \cos \frac{1}{2}d (\sin \frac{1}{2}a \sin \frac{1}{2}b + \sin \frac{1}{2}c \sin \frac{1}{2}d)}; \tag{4}$$

and since (Art. 33)

$$\cos^2 \tfrac{1}{2}\widehat{ab} = \frac{\sin^2 \frac{1}{2}(a+b) - \sin^2 \frac{1}{2}\delta}{\sin a \sin b},$$

we get, by substituting for $\sin^2 \frac{1}{2}\delta$, in terms of the sides, and reducing,

$$\cos^2 \tfrac{1}{2}\widehat{ab} = \frac{\cos \frac{1}{4}(\overline{a+b} + \overline{c+d}) \cos \frac{1}{4}(\overline{a+b} - \overline{c+d}) \sin \frac{1}{2}(s-c) \sin \frac{1}{2}(s-d)}{\cos \frac{1}{2}a \cos \frac{1}{2}b (\sin \frac{1}{2}a \sin \frac{1}{2}b + \sin \frac{1}{2}c \sin \frac{1}{2}d)} \tag{5}$$

and

$$\cos^2 \tfrac{1}{2}\widehat{cd} = \frac{\cos \frac{1}{4}(\overline{a+b} + \overline{c+d}) \cos \frac{1}{4}(\overline{a+b} - \overline{c+d}) \sin \frac{1}{2}(s-a) \sin \frac{1}{2}(s-b)}{\cos \frac{1}{2}c \cos \frac{1}{2}d (\sin \frac{1}{2}a \sin \frac{1}{2}b + \sin \frac{1}{2}c \sin \frac{1}{2}d)}. \tag{6}$$

---

* A cyclic spherical quadrilateral can thus be constructed when the four sides are known.

From (3) and (5) we get

$$\tan^2 \tfrac{1}{2}\,\widehat{ab} = \frac{\cos\tfrac{1}{4}(\overline{a-b}+\overline{c-d})\cos\tfrac{1}{4}(\overline{a-b}-\overline{c-d})\sin\tfrac{1}{2}(s-a)\sin\tfrac{1}{2}(s-b)}{\cos\tfrac{1}{4}(\overline{a+b}+\overline{c+d})\cos\tfrac{1}{4}(\overline{a+b}-\overline{c+d})\sin\tfrac{1}{2}(s-c)\sin\tfrac{1}{2}(s-d)}.$$

Prop. 5.—**To find the area of a spherical quadrilateral inscribed in a circle, in terms of the sides.**

From Art. 111, *Cor.* 2, it follows, that if E denote the excess of the quadrilateral,

$$\sin\tfrac{1}{4}E = -\cos\tfrac{1}{2}(\widehat{ab}+\widehat{cd}),$$

expanding the right member of this equation, and substituting the values obtained for $\sin\tfrac{1}{2}\widehat{ab}$, &c., in the previous Proposition ; on reduction we find

$$\sin\tfrac{1}{4}E = \sqrt{\frac{\sin\tfrac{1}{2}(s-a)\sin\tfrac{1}{2}(s-b)\sin\tfrac{1}{2}(s-c)\sin\tfrac{1}{2}(s-d)}{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\tfrac{1}{2}d}}. \qquad (1)$$

By a similar method we obtain

$$\cos\tfrac{1}{4}E = \sqrt{\frac{\cos\tfrac{1}{4}(\overline{a+b}+\overline{c+d})\cos\tfrac{1}{4}(\overline{a+b}-\overline{c+d})\cos\tfrac{1}{4}(\overline{a-b}+\overline{c-d})\cos\tfrac{1}{4}(\overline{a-b}-\overline{c-d})}{\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\tfrac{1}{2}d}}. \qquad (2)$$

Hence, dividing (1) by (2),

$$\tan\tfrac{1}{4}E = \sqrt{\frac{\sin\tfrac{1}{2}(s-a)\sin\tfrac{1}{2}(s-b)\sin\tfrac{1}{2}(s-c)\sin\tfrac{1}{2}(s-d)}{\cos\tfrac{1}{4}(\overline{a+b}+\overline{c+d})(\,.\,.\,.\,.\,)(\,.\,.\,.\,)(\,.\,.\,.\,.\,)}}. \qquad (3)$$

Prop. 6.—**To find the diagonal of a parallelepiped in terms of its edges, and their mutual inclinations.**

Let $OA = x$, $OB = y$, $OC = z$, and let the inclinations be $BOC = \alpha$, $COA = \beta$, $AOB = \gamma$. Then the triangle OED gives

$$OD^2 = OE^2 + z^2 + 2z\,.\,OE\cos COE$$

$$= x^2 + y^2 + z^2 + 2xy\cos\gamma + 2z\,OE\cos COE.$$

Now, it is clear that $OE\cos COE$ is the projection of OE on the line OC, and therefore it must be equal to the sum of the projections of OB and BE on the same line, or of OB and OA, that is

$$OE \cos COE = y \cos \alpha + x \cos \beta.$$

Hence

$$OD^2 = x^2 + y^2 + z^2 + 2yz \cos \alpha + 2zx \cos \beta + 2xy \cos \gamma.$$

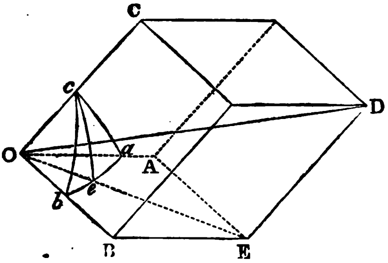

Fig. 73.

The same result may be arrived at by describing a sphere round O as centre, meeting the lines OA, OB, OC, OE in a, b, c, e, respectively, and by determining the angle COE, that is, the arc ce, by aid of Art. 40, and by noticing that $OE \sin AOE = y \sin \gamma$, and $OE \sin BOE = x \sin \gamma$.

Prop. 7.—**Volume of Parallelepiped.**—The volume of the parallelepiped (fig. 73) is equal to the area of the face $OAEB$ multiplied by the perpendicular distance of C from it, that is

$$\text{volume} = \tfrac{1}{2}\, xy \sin \widehat{ab} \times z \sin p,$$

where p is the perpendicular arc from c on ab; therefore

$$\text{volume} = \tfrac{1}{2}\, xyz\ (n \text{ of triangle } abc)$$
$$= xyz\ (1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma)^{\frac{1}{2}}.$$

Prop. 8.—**If S denote the number of solid angles of any polyhedron, F the number of its faces, and E the number of its edges, to show that**

$$S + F = E + 2.$$

Let all the edges of the polyhedron be projected on a sphere of unit radius, whose centre lies within the polyhedron. Each face will thus project into a spherical polygon [Art. 177 (γ)]. Now, if s denote the sum of the angles of one of the polygons, and n the number of its sides, its area is equal to $s - (n - 2)\pi$; therefore, since the sphere is divided into F polygons, its area, or $4\pi = \Sigma s - \pi\Sigma n + 2\pi F$; but $\Sigma s = 2\pi S$, and $\Sigma n = 2E$, since every arc on the sphere is a common side of two polygons.

Substituting these values, we have

$$4\pi = 2\pi S - 2\pi E + 2\pi F;$$

or

$$S + F = E + 2.$$

Prop. 9.—**If a spherical diameter HC of a small circle be drawn through the vertex C of an inscribed triangle to meet the base in P; prove the relation**

$$\tan \tfrac{1}{2} AHP \tan \tfrac{1}{2} BHP = \frac{\sin (R - \delta)}{\sin (R + \delta)},$$

where $\delta = HP$.

Applying Napier's Analogies, Art. 56 (1), to the triangle AHP, and Art. 56 (2) to the triangle BHP, and multiplying the results together, the above equation is obtained.

Cor.—If C is a right angle, $\cos^2 R = \dfrac{\sin (R - \delta)}{\sin (R + \delta)}$.

[Apply Art. 44 (5) to the triangles HCX and HCY (fig. 32)].

Hence—*If a variable right-angled triangle, with fixed vertex, be inscribed in a small circle, its base passes through a fixed point;* and reciprocally, *if the base of a variable triangle, whose inscribed circle is given fixed in position, be equal to a quadrant, the locus of the vertex is a great circle.*

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THE END.

EXAMINATION PAPERS.

[The following Examples have been taken chiefly from University Calendars, Science and Art Examination Papers, and Mathematical Journals. They are arranged in sets, without any attempt at graduation. Answers are given in almost all cases, and hints to the solution are given where any difficulty is presented.]

I.

1. Being given of a triangle the base and difference of base angles, and that the sum of its sides is 180°, construct it.

2. Prove the following formula for an equilateral triangle:—

$$\log \sin \frac{A}{2} + \log \cos \frac{a}{2} + \log 2 = 0.$$

3. Find $\cos x$, in terms of a and b, from the equations

$$\begin{aligned} \cos a &= \cos \theta \cos x, \\ \cos b &= \cos \phi \cos x, \\ \cos (\theta + \phi) &= \cos a \cos b. \end{aligned}$$

(*Science and Art Hon. Exam.*)

4. Given the base, sum of sides, and one base angle; find the other parts of the triangle. (*Science and Art Hon. Exam.*)

5. If an angle of a triangle be equal to (or supplemental to) the opposite side, show that

$$1 - \sec^2 a - \sec^2 b - \sec^2 c + 2 \sec a \sec b \sec c = 0.$$

6. Find the locus of the intersection of equal tangents to two small circles.

7. If each leg of a tripod be inclined at an angle ϕ to each of the other legs; find the angle between a leg and the horizontal plane, the legs being supposed equal to each other.

8. Given that $\tan x = \tan a \cos b$,

and $\tan y = \tan b \cos a$;

show that $\sec (x \pm y) = \sec a \sec b \pm \tan a \tan b$.

(*Science and Art Hon. Exam.*)

9. Given $a = 84° \ 30'$, $b = 46° \ 10'$, $C = 75° \ 46'$;
calculate A, B, and c.

10. Given $a = 86° \ 45'$, $b = 74° \ 35'$, $c = 63° \ 40'$;
calculate A, B, and C.

II.

1. Given the base of c of a triangle, and that

$$\tan \tfrac{1}{2} a \tan \tfrac{1}{2} b = \tan^2 \tfrac{1}{2} \beta ;$$

find $a - b$, β being the bisector of the base.

2. If $C = A + B$, show that

$$1 - \cos a - \cos b + \cos c = 0.$$

3. If A denote one of the angles of an equilateral triangle, and A' an angle of its polar triangle, show that

$$\cos A \cos A' = \cos A + \cos A'.$$

Explain the result when $A = 60°$.

(*Science and Art Exam. Papers.*)

4. Express in terms of C and c.

$$\frac{\cos a \cos B - \cos b \cos A}{\sin a - \sin b}.$$

5. If four points, A, B, C, D on a great circle be joined to a point P on the sphere, show that

$$\begin{vmatrix} \cot AB & \cot APB & 1 \\ \cot AC & \cot APC & 1 \\ \cot AD & \cot APD & 1 \end{vmatrix} = 0.$$

6. Using the notation of Ex. 5, show that if $\cot AB$, $\cot AC$, $\cot AD$, are in Arithmetical Progression, $\cot APB$, $\cot APC$, $\cot APD$, are also in Arithmetical Progression, and hence *a harmonic row always subtends a harmonic pencil at any point on the sphere.*

7. The sum of the sides of a spherical polygon is less than 2π.

8. Prove the following formula for the reduction of the parts of a spherical triangle :—

$$(\sec a \sin b \cos A - \sin c)^2 + (\sec a \cos b - \cos c)^2 (1 - \operatorname{cosec}^2 a \sin^2 A)$$
$$= \tan^2 a \cos^2 B \cos^2 C.$$

(*Educational Times*, vol. xxxvii. p. 69.)

9. Given $a = 72° \, 44'$, $b = 36° \, 22'$, $C = 56° \, 12'$; calculate A, B, and c.

10. Given, in a right-angled triangle, $a = 86° \, 45'$, $b = 108° \, 20'$; calculate A, B, and c.

III.

1. If two small circles on a sphere touch each other, show that the great circle joining their poles passes through their point of contact.

2. Prove that

$$\cos A = \frac{\cos a \sin b - \sin a \cos b \cos C}{\sin c};$$

and hence

$$\cos A + \cos B = \frac{2 \sin (a+b) \sin^2 \frac{1}{2} C}{\sin c}.$$

(Cf. Art. 35, Ex. 1.)

3. Show that the arcs joining the middle points of the sides of the colunar triangles of a given spherical triangle are equal to the corresponding angles of the chordal triangle.

4. Prove Legendre's Theorem by means of the relations

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

5. Find the locus of the middle points of the sides of a triangle, having given the base and the sum of the three angles.

6. Show that

$$\Sigma \sin^2 a \cot XY = 0,$$

where XY is the intercept made by the bisectors of the angle A on the side a of a spherical triangle.

7. Deduce the analogue *in plano* of Ex. 6.

8. Having given the sides of a triangle, each $= 60°$; find the segments of the bisectors of the angles.

9. In a right-angled triangle, being given

$$A = 47° \ 45', \quad B = 63° \ 30';$$

calculate the sides.

10. Given $a = 75° \ 12', \quad b = 63° \ 27', \quad c = 52° \ 23';$
calculate A, B, and C.

IV.

1. Having given the relation

$$\tan \tfrac{1}{2}a \tan \tfrac{1}{2}b = \tan^2 \tfrac{1}{2}c;$$

show that

$$\cos \beta = \frac{1 - \sin^2 c \cos^2 \theta}{1 - \sin^2 c \sin^2 \theta},$$

where β is the bisector of the side $2c$ drawn from the opposite vertex, and θ the angle it makes with that side.

(*Dublin Univ. Exam. Papers.*)

2. Show that the sum of the three arcs joining the middle points of the sides of the three colunar triangles of a given triangle is equal to two right angles.

3. If

$$\sin^2 \frac{c}{2} = \sin^2 \frac{a}{2} + \sin^2 \frac{b}{2},$$

prove that $C = A + B$.

4. Given the hypotenuse of a right-angled triangle; prove that the difference of the sides is a maximum when their sum is equal to a quadrant.

5. Find the relation connecting the angles of a triangle if

$$1 + \cos a + \cos b + \cos c = 0.$$

6. Prove that

$$\frac{\cos (B - C)}{\cos (A - C)} = \frac{\tan \frac{1}{2}a - \tan \frac{1}{2}b \cos C - \tan \frac{1}{2}c \cos B}{\tan \frac{1}{2}b - \tan \frac{1}{2}a \cos C - \tan \frac{1}{2}c \cos A}.$$

7. The great circle through the poles of two small circles is perpendicular to the great circle through their points of intersection.

8. If α and α' are the segments of the bisector of the side a of a triangle, show that

$$\frac{\sin \alpha}{\sin \alpha'} = 2 \cos \frac{a}{2}.$$

9. Given $A = 58° \; 16'$, $B = 41° \; 28'$, $c = 75° \; 54'$; calculate a, b, and C.

10. Given $a = 47° \; 45'$, $c = 69° \; 30'$, $C = 90°$; calculate A, B, and b.

V.

1. If a, b, c, d be the sides of a spherical quadrilateral, δ and δ' its diagonals, and if a and c, b and d, δ and δ', intersect at angles θ, ϕ, ψ, respectively, show that

$$\sin a \sin c \cos \theta + \sin b \sin d \cos \phi + \sin \delta \sin \delta' \cos \psi = 0.$$

2. Find the locus of a point P, whose distances from three fixed points A, B, C, are connected by the relation

$$\cos AP + \cos BP + \cos CP = \text{const.}$$

3. If α, α'; β, β'; γ, γ', be the segments of the perpendiculars to the sides of a spherical triangle drawn from the opposite vertices, show that

$$\tan \alpha \tan \alpha' = \tan \beta \tan \beta' = \tan \gamma \tan \gamma'.$$

4. Using the notation of Ex. 3, show that

$$\frac{\cos(\alpha + \alpha')}{\cos \alpha \cos \alpha'} = \frac{\cos(\beta + \beta')}{\cos \beta \cos \beta'} = \frac{\cos(\gamma + \gamma')}{\cos \gamma \cos \gamma'}.$$

5. Simplify the expression

$$\frac{\cos A \cot a + \cos B \cot b}{\cot a \cot b - \cos A \cos B}.$$

6. If the bisectors of the angles of a triangle meet at O, show that the angle AOB is supplemental to the angle COD, where OD is a perpendicular from O on BC.

7. Using the notation of Ex. 6, show that—

(1) $\tan AO \cos BOC = \tan BO \cos COA = \tan CO \cos AOB$;

(2) $\sin BOC : \sin COA : \sin AOB = \cos \frac{1}{2} A : \cos \frac{1}{2} B : \cos \frac{1}{2} C$.

8. The angle in a semicircle is a right angle; what is the analogous theorem on the sphere?

9. Given $a = 85^\circ\ 16'$, $b = 63^\circ\ 24'$, $C = 76^\circ\ 12'$; calculate A, B, and c.

10. In a right-angled triangle, given $c = 48^\circ\ 45'$, and $A = 21^\circ\ 27'$: calculate a, B, and b.

VI.

1. Two places are situated on the same parallel of latitude λ; find the difference of the distances sailed over by two ships passing between them, one keeping to the great circle course, the other to the parallel; the difference of longitude of the places being $2l$.

2. Having given the base c of a spherical triangle, and the ratio $\sin \frac{1}{2}a : \sin \frac{1}{2}b$, find the locus of the vertex.

3. If α, β, γ, be the sides of the triangle formed by joining the middle points of the sides b, c; c, a; a, b, respectively, of a given triangle, show that

$$\sin \alpha : \sin \beta : \sin \gamma :: \operatorname{cosec} \widehat{\alpha a} : \operatorname{cosec} \widehat{\beta b} : \operatorname{cosec} \widehat{\gamma c}.$$

4. Using the notation of the preceding Example, show that

$$\tan \widehat{\alpha a} \sin \tfrac{1}{2}a = \tan \widehat{\beta b} \sin \tfrac{1}{2}b = \tan \widehat{\gamma c} \sin \tfrac{1}{2}c.$$

5. If the sides of a spherical triangle be each 60°; show that the circles described, each having a vertex for pole, and passing through the middle points of the sides which meet at it, have the sides of the supplemental triangle for common tangents.

6. If the bisectors of the angles of a triangle meet at O, show that

$$\frac{\sin BOC}{\sin AO} : \frac{\sin COA}{\sin BO} : \frac{\sin AOB}{\sin CO} = \sin a : \sin b : \sin c.$$

7. Using the notation of Ex. 6; show that

$$\sin^2 AO : \sin^2 BO : \sin^2 CO = \frac{\sin (s-a)}{\sin a} : \frac{\sin (s-b)}{\sin b} : \frac{\sin (s-c)}{\sin c}.$$

8. If the base of a triangle be divided into four equal parts, which subtend angles, θ_1, θ_2, θ_3, θ_4, at the vertex; show that

$$\frac{\sin (\theta_1 + \theta_2)}{\sin (\theta_3 + \theta_4)} = \frac{\sin \theta_1 \sin \theta_3}{\sin \theta_2 \sin \theta_4}.$$

9. Given $A = 86° \, 44'$, $B = 58° \, 32'$, $c = 79° \, 40'$;
calculate a, b, and C.

10. Given $A = 46° \, 45'$, $c = 75° \, 40'$, $C = 90°$;
calculate a, b, and B.

VII.

1. If a great circle cuts a fixed great circle at a constant angle ω, find the locus of its pole.

2. Show from Ex. 1, Paper v., that the perpendiculars of a spherical triangle are concurrent.

3. Given three fixed points, A, B, C; find the locus of a fourth point P on the sphere, such that

$$l \cos AP + m \cos BP + n \cos CP = \text{const.},$$

where l, m, n are given quantities.

4. If β be the bisector of the base c of a triangle, show that

$$\sin \beta = \frac{\sqrt{\sin^2 a + \sin^2 b + 2 \sin a \sin b \cos C}}{2 \cos \frac{1}{2} c}.$$

5. Deduce the corresponding equation for a plane triangle.

6. A spherical quadrilateral is circumscribed to a small circle; show that the sum of one pair of opposite sides is equal to the sum of the other pair.

7. In a spherical triangle, prove that—

(1) If three sides are acute, two angles are acute;

(2) If one side is acute and one side is not acute, one angle is obtuse and two are acute;

(3) If two sides are obtuse and one acute, one angle is obtuse;

(4) If two sides are obtuse and the other is not acute, all the angles are obtuse.

(*Educational Times.*)

8. Prove the relations—

(1) $\cos^2 \frac{1}{2} c = \cos^2 \frac{1}{2} (a - b) \cos^2 \frac{1}{2} C + \cos^2 \frac{1}{2} (a + b) \sin^2 \frac{1}{2} C$;

(2) $\sin^2 \frac{1}{2} c = \sin^2 \frac{1}{2} (a - b) \cos^2 \frac{1}{2} C + \sin^2 \frac{1}{2} (a + b) \sin^2 \frac{1}{2} C$.

9. Given $a = 68° \ 16'$, $b = 53° \ 24'$, $C = 86° \ 40'$; calculate A, B, and c.

10. Given $a = 38° \ 40'$, $c = 73° \ 30'$, $C = 90°$; calculate A, B, and b.

VIII.

1. From a point P on a small circle, in which is inscribed an equilateral triangle ABC, arcs of great circles PA, PB, PC are drawn; prove that

$$\sin \tfrac{1}{2} AP + \sin \tfrac{1}{2} BP = \sin \tfrac{1}{2} CP,$$

and give the corresponding property *in plano*.

2. Prove *Legendre's Theorem* from the formula

$$\cos^2 \tfrac{1}{2} A = \frac{\sin s\,(s-a)}{\sin b \sin c}.$$

3. Prove *Legendre's Theorem* from each of the formulæ for $\sin \frac{1}{2} A$ and $\tan \frac{1}{2} A$, respectively, in terms of the sides.

4. Deduce the area of a plane triangle, in terms of the base and base angles, from the formulæ.

$$\sin^2 \frac{a}{2} = \frac{-\cos S \cos (S-A)}{\sin B \sin C}, \text{ \&c., \&c.}$$

5. From the *rule of sines*, Art. 31, find the area of a plane triangle in terms of the base and base angles.

6. Prove directly, employing the solution in Art. 29, that

$$\frac{\tan^2 \frac{A}{2} - \tan^2 \frac{A'}{2}}{\tan^2 \frac{A'}{2}} = \frac{bc}{3r^2}, \text{ nearly.}$$

7. If A and A', B and B', C and C', denote the pairs of opposite angles of a complete quadrilateral, prove the relation

$$\cos B \cos B' \sim \cos C \cos C' = \sin A \sin A' \cos AA'.$$

8. Using the notation of Ex. 7, show that for a plane quadrilateral—

$$(\alpha)\quad \cos B \cos B' \sim \cos C \cos C' = \sin A \sin A';$$

$$(\beta)\quad \sin A \sin A' \sim \sin B \sin B' = \sin C \sin C'.$$

9. Given $a = 87° \ 27'$, $b = 63° \ 19'$, $c = 74° \ 40'$;
calculate A, B, and C.

10. Given $a = 38° \ 50'$, $B = 47° \ 44'$, $C = 90°$;
calculate A, b, and c.

IX.

1. If A and A', B and B', C and C', denote the pairs of opposite angles of a quadrilateral, and δ_1, δ_2, δ_3, the diagonals, prove that the (*algebraic*) sum

$$\sin A \sin A' \cos \delta_1 + \sin B \sin B' \cos \delta_2 + \sin C \sin C' \cos \delta_3 = 0.$$

2. If two diagonals of a spherical quadrilateral are each quadrants, the third diagonal is also a quadrant.

(*Dub. Univ. Exam. Papers.*)

3. The intersections of the corresponding sides of a triangle and its polar triangle lie on a great circle.

4. Prove generally that if three great circles be drawn from the vertices of a triangle ABC through any point O, intersecting the opposite sides of the triangle respectively at angles θ, ϕ, ψ,

$$\sin a \sin AO \cos \theta + \sin b \sin BO \cos \phi + \sin c \sin CO \cos \psi = 0.$$

5. In a plane triangle ABC three right lines, AP, BP, CP, are drawn, cutting the opposite sides respectively in X, Y, Z; show that

$$a\, AP \cos X + b\, BP \cos Y + c\, CP \cos Z = 0.$$

6. Using the notation of Ex. 4 for an equilateral spherical triangle, show that

$$\sin AO \cos \theta + \sin BO \cos \phi + \sin CO \cos \psi = 0.$$

7. If ϕ denote the perpendicular from the angle A on the opposite side a, and β and γ the segments of the base made by it, prove the relation

$$\sin^2 p \cos \beta \cos \gamma - \sin p \sin (\beta + \gamma) \cot A - \sin \beta \sin \gamma = 0;$$

and hence show that of all spherical triangles described on the same base, and having equal altitudes, the vertical angle is a maximum when the triangle is isosceles.—(W. Nicolls.)

8. If a, b, c, d, be the sides of a cyclic spherical quadrilateral, δ, one of the diagonals, is given by the equation

$$\sin^2 \tfrac{1}{2} \delta = \frac{(\sin \frac{1}{2} a \sin \frac{1}{2} d + \sin \frac{1}{2} b \sin \frac{1}{2} c)(\sin \frac{1}{2} a \sin \frac{1}{2} c + \sin \frac{1}{2} b \sin \frac{1}{2} d)}{\sin \frac{1}{2} a \sin \frac{1}{2} b + \sin \frac{1}{2} c \sin \frac{1}{2} d}.$$

9. Given $A = 69° 56'$, $B = 46° 36'$, $c = 76° 28'$; calculate a, b, and C.

10. Given $a = 45° 30'$, $B = 63° 45'$, $C = 90°$; calculate A, b, and c.

X.

1. Given $A + a$, $B + b$, and $C + c$; solve the triangle.

(*Dub. Univ. Exam. Papers.*)

2. If O be the mid-point of an equilateral triangle ABC, and P any point on the surface of the sphere, show that

$$(\tan PO \tan AO)^2 (\cos AP + \cos BP + \cos CP)^2$$
$$= 4 [\cos^2 AP + \cos^2 BP + \cos^2 CP - \cos BP \cos CP - \cos CP \cos AP - \cos AP \cos BP].$$

(*Cambridge Univ. Exam. Papers.*)

3. The perpendiculars of a triangle are concurrent; hence show from the supplemental figure, that if two diagonals of a quadrilateral are quadrants, the third is also a quadrant.

4. Using the notation of Ex. 2, and each side of the triangle being a quadrant, show that

$$\cos AP + \cos BP + \cos CP = \sqrt{3} \cos OP.$$

5. Construct a triangle right-angled at C, being given the angle A and the sum or difference of the sides which contain it.

6. Given the base of a spherical triangle, and the length of the arc joining the middle points of the sides; find the locus of the vertex.

7. If the angle A of a right-angled triangle be given, show that the difference of the sides which contain it is a maximum when their sum is a quadrant.

8. Given the base of a triangle, and the locus of the vertex a great circle; construct the triangle, having a maximum or minimum sum of sides.

9. Given $a = 64° 30'$, $b = 48° 12'$, $C = 90$;

calculate the remaining parts.

10. Solve the triangle when

$$a = 56° 48', \quad b = 42° 26', \quad C = 83° 12'.$$

XI*.

1. Find the locus of a point P, such that

$$\sin PX + \sin PY = \text{const.},$$

where PX and PY are secondaries to two given great circles.

2. Find the locus of a point P, such that

$$l \sin PX \pm m \sin PY = \text{const.},$$

where PX and PY are secondaries to two given great circles, and l and m constants.

3. Find the locus of a point P, such that

$$(\alpha)\ \sin PX + \sin PY + \sin PZ = \text{const.};$$
$$(\beta)\ l \sin PX \pm m \sin PY \pm n \sin PZ = \text{const.},$$

where X, Y, Z, are the feet of the perpendiculars from P on the sides of a spherical triangle.

4. If equation (1), Art. 43, be written in the form

$$at^2 - 2bt \cos\theta + c = 0,$$

prove that it will represent a great circle if $a + c = 0$.

5. Construct a right-angled triangle, having given the hypotenuse, and the locus of the vertex a given great circle.

6. Having given the base and the sum of the sides, show that the product of the sines of the perpendiculars from the extremities of the base on the bisector of the external vertical angle is constant.

7. Having given the base and the difference of the sides, show that the product of the sines of the perpendiculars from the extremities of the base on the bisector of the internal vertical angle is constant.

8. If p_1 and p_2 denote the perpendiculars from the base angles on the bisector of the external vertical angle, and p_3 and p_4 those on the bisector of the internal vertical angle, prove the relation

$$\sin p_1 \sin p_2 + \sin p_3 \sin p_4 = \sin a \sin b.$$

9. Solve the triangle, having given $A = 29^\circ\, 33'$, $B = 87^\circ\, 21'$, and $C = 90^\circ$.

10. Having given $a = 62^\circ\, 56'$, $b = 39^\circ\, 28'$, $C = 75^\circ\, 18'$, solve the triangle.

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[*] Part of this Paper and part of the preceding were communicated to us by Dr. Traill, F.T.C.D.

XII.*

1. If β be the bisector of the base AB of a triangle ABC, and if β' be the bisector of the base BA' of the colunar BCA', show that

$$(\alpha)\quad \sin^2\beta\cos^2\frac{c}{2} - \sin^2\beta'\sin^2\frac{c}{2} = \sin a\sin b\cos C;$$

$$(\beta)\quad \sin^2\beta\cos^2\frac{c}{2} + \sin^2\beta'\sin^2\frac{c}{2} = \tfrac{1}{2}(\sin^2 a + \sin^2 b).$$

2. Show that if $E = A + B + C - \pi$, then

$$\sin\tfrac{1}{2}E\sin(A - \tfrac{1}{2}E)\sin(B - \tfrac{1}{2}E)\sin(C - \tfrac{1}{2}E) = N^2.$$

3. Show that

$$\tan^2\frac{c}{2} = \frac{\tan^2\frac{1}{2}a - 2\tan\frac{1}{2}a\tan\frac{1}{2}b\cos C + \tan^2\frac{1}{2}b}{1 + 2\tan\frac{1}{2}a\tan\frac{1}{2}b\cos C + \tan^2\frac{1}{2}a\tan^2\frac{1}{2}b}.$$

4. Show that

$$\Sigma\tan\frac{a}{2}\,\frac{\sin\frac{1}{2}(B - C)}{\sin\frac{1}{2}B\sin\frac{1}{2}C} = 0.$$

5. If the base AB of a triangle be divided into segments α and β by an arc θ drawn from the vertex C, show that

$$(\alpha)\quad \cos\theta\sin c = \cos a\sin\alpha + \cos b\sin\beta.$$

If θ be the bisector of the vertical angle, show that

$$(\beta)\quad \cos\theta = \frac{\sin(a + b)}{\sin a\cos\alpha + \sin b\cos\beta}.$$

6. If the internal bisector of the angle C of a triangle meets the opposite side at an angle θ, and divides it into segments α and β adjoining the angles A and B, respectively, and if the external bisector of C makes an angle θ' with the opposite side, show that

$$(1)\quad \sin\alpha\sin\beta = \frac{\sin a\sin b\sin^2 c}{\sin^2 a + \sin^2 b + 2\sin a\sin b\cos c}.$$

$$(2)\quad \sin\theta = \frac{\sin\frac{1}{2}C}{\sin c}\sqrt{\sin^2 a + \sin^2 b + 2\sin a\sin b\cos c}.$$

$$(3)\quad \cos\theta\cos\theta' = \frac{\sin(A + B)\sin(A - B)}{2\sin C}.$$

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* The greater part of this Paper was communicated to us by Mr. W. S. M'Cay, F.T.C.D.

7. Being given the angle A of a right-angled triangle, show that when $c - b$ is a maximum,

$$\cos A = \cos^2 a.$$

8. Given $A = 56° 58'$, $B = 39° 26'$, $c = 68° 48'$;
calculate the remaining parts.

9. Given $A = 54° 40'$, $B = 75° 20'$, $C = 62° 44'$;
solve the triangle.

10. Given $a = 84° 30'$, $b = 56° 46'$, $C = 73° 52'$;
calculate the remaining parts.

11. Given $a = 40° 45'$. $A = 51° 50'$, $C = 90°$;
solve the triangle.

12. Given $a = 51° 20'$, $A = 62° 12'$, $C = 90°$;
solve the triangle.

13. Given $A = 81° 54'$, $B = 67° 12'$, $c = 87° 42'$;
solve the triangle.

14. Given $A = 67° 37'$, $B = 39° 19'$, $c = 84° 46'$;
solve the triangle.

15. Given $A = 63° 40'$, $B = 87° 10'$, $C = 59° 46'$;
solve the triangle.

ANSWERS TO EXAMINATION PAPERS.

I.

1. The sum of the base angles is given by Art. 24.

2. Apply formula for $\sin \frac{1}{2} A$ (Art. 33).

3. x is the perpendicular from the vertex on the hypotenuse of a right-angled triangle, the sides of which are a and b, and the hypotenuse $\theta + \phi$; hence

$$\sin x = \frac{\sin a \sin b}{\sin (\theta + \phi)}. \qquad \text{[Art. 36, Ex. 2.]}$$

4. The remaining parts may be determined by aid of *Napier's Analogies*, or geometrically from fig. 22.

5. Suppose $a = A$; then, from Art. 20,

$$\sec a = \sec b \sec c - \tan b \tan c;$$

and therefore

$$\tan^2 b \tan^2 c = (\sec a - \sec b \sec c)^2;$$

therefore, &c.

6. A great circle perpendicular to that through the poles of the small circles, and dividing it so that the ratio of the cosines of its segments is equal to the ratio of the cosines of radii of the small circles—in other words, the great circle through the intersections of the small circles.

7. If θ be the inclination to the horizon,

$$\cos \theta = \frac{2}{\sqrt{3}} \sin \frac{\phi}{2}.$$

8. x and y are the segments of the base of a triangle made by the perpendicular from the vertex, the sides a and b being equal to the opposite angles; therefore, &c.

9. $A = 95° \; 56' \; 17{\cdot}87''$, $B = 46° \; 7' \; 15{\cdot}53''$, $c = 75° \; 56' \; 27''$.

10. $A = 94° \; 3' \; 50{\cdot}28''$, $B = 74° \; 23' \; 45{\cdot}16''$, $C = 63° \; 33' \; 42{\cdot}48''$.

II.

1. We have

$$\frac{1-\tan\frac{1}{2}a\tan\frac{1}{2}b}{1+\tan\frac{1}{2}a\tan\frac{1}{2}b}=\frac{1-\tan^2\beta}{1+\tan^2\beta}.$$

Hence

$$\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\cos\beta=\frac{\cos a+\cos b}{2\cos\frac{1}{2}c}; \qquad \text{(Art. 42.)}$$

and therefore $$\cos^2\tfrac{1}{2}(a-b)=\cos\tfrac{1}{2}c.$$

2. Use *Delambre's Analogies.* See also Art. 43, Ex. 29.

3. See Art. 27, Ex. 8.

4. Substitute for $\cos A$ and $\cos B$ in terms of the sides, and the expression reduces to

$$\frac{\cos C+\cos c}{\sin c}.$$

5. Apply the formulae of Art. 37 to the triangles *APB*, *APC*, *APD*, and eliminate $\sin A$ and $\cos A$.

6. This appears at once from the determinant of Ex. 5.

7. See Art. 19 (1).

8. $(\sec a\sin b\cos A-\sin c)^2+(\sec a\cos b-\cos c)^2\,(1-\operatorname{cosec}^2 a\sin^2 A)$

$$=\left(\frac{\sin b\cos A-\cos a\sin c}{\cos a}\right)^2+\left(\frac{\cos b-\cos a\cos c}{\cos a}\right)^2\left(1-\frac{\sin^2 A}{\sin^2 a}\right)$$

$$=\left(\frac{\sin b\sin c\cos A-\cos a\sin^2 c}{\cos a\sin c}\right)^2+\left(\frac{\sin^2 c\sin^2 a\cos^2 B}{\cos^2 a}\right)\left(1-\frac{\sin^2 A}{\sin^2 a}\right)$$

$$=\left(\frac{\cos a-\cos b\cos c-\cos a\sin^2 c}{\cos a\sin c}\right)^2+\tan^2 a\cos^2 B\,(\sin^2 c-\sin^2 C)$$

$$=\frac{\cos^2 c\,(\cos a\cos c-\cos b)^2}{\sin^2 c\cos^2 a}+\tan^2 a\cos^2 B\,(\sin^2 c-\sin^2 C)$$

$$=\cos^2 c\tan^2 a\cos^2 B+\tan^2 a\cos^2 B\,(\sin^2 c-\sin^2 C)$$
$$=\tan^2 a\cos^2 B\cos^2 C.$$

9. $A=107^\circ\ 36'\ 9{\cdot}4''$, $B=36^\circ\ 17'\ 20{\cdot}6''$, $c=56^\circ\ 21'\ 30''$.

10. $A=86^\circ\ 54'\ 52{\cdot}7''$, $B=108^\circ\ 18'\ 20{\cdot}9''$, $c=101^\circ\ 1'\ 18{\cdot}5''$.

III.

1. The great circle is the shortest distance between the poles. See Art. 3.

2. In the value of $\cos A$ [Art. 26 (2)] substitute the value of $\cos c$ [Art. 26 (1)], and reduce.

3. By Ex. 14, Art. 12, the radii of the sphere drawn to the middle points of the sides of a colunar are parallel to the sides of the chordal triangle; and hence the angle between them, which is the arc joining the middle points of the sides of the colunar, is equal to the corresponding angle of the chordal triangle.

4. Let A', B', C' be the angles, and α, β, γ the sides of the plane triangle. (See Art. 29.)

Then $$\sin A = \frac{\sin a}{\sin b} \sin B, \quad \text{and} \quad \sin A' = \frac{\alpha}{\beta} \sin B',$$

also $$\frac{\sin a}{\sin b} = \frac{\alpha}{\beta}, \text{ nearly};$$

hence we have $$\sin A + \sin A' = \frac{\alpha}{\beta} (\sin B + \sin B'),$$

or $$\sin \tfrac{1}{2}(A + A') \cos \tfrac{1}{2}(A - A') = \frac{\alpha}{\beta} \sin \tfrac{1}{2}(B + B') \cos \tfrac{1}{2}(B - B').$$

Now take $\sin \frac{1}{2}(A + A')$ and $\sin \frac{1}{2}(B + B')$, respectively equal to $\sin A$ and $\sin B$; and, making these substitutions in the preceding equation, we have

$$\sin A \cos \tfrac{1}{2}(A - A') = \frac{\alpha}{\beta} \sin B \cos \tfrac{1}{2}(B - B');$$

therefore

$$\cos \tfrac{1}{2}(A - A') = \cos \tfrac{1}{2}(B - B') = \cos \tfrac{1}{2}(C - C').$$

Now $$A - A', \quad B - B', \quad C - C' \quad \text{are each} \quad < \tfrac{1}{2}\pi$$

hence $$A - A' = B - B' = C - C' = \tfrac{1}{3}(A + B + C - \pi);$$

therefore, &c.

5. The locus is the circle LM, fig. 10.

6. This follows from the value of $\cot XY$ given, Art. 39, Ex. 6.

7. See Casey's "Sequel to Euclid," sec. VIII. Ex. 17—

$$\frac{a^2}{XY} + \frac{b^2}{X'Y'} + \frac{c^2}{X''Y''} = 0.$$

8. If α and α' are the segments, we have

$$\cos(\alpha + \alpha') = \frac{1}{\sqrt{3}}, \text{ and } \sin\alpha = \sqrt{3}\sin\alpha';$$

therefore, &c.

9. $a = 41° \ 17' \ 48''$, $b = 52° \ 55' \ 47{\cdot}5''$, $c = 63° \ 4' \ 17''$.

10. $A = 91° \ 24' \ 19{\cdot}5''$, $B = 67° \ 39' \ 45{\cdot}8''$, $C = 54° \ 59' \ 22{\cdot}26''$.

IV.

1. Using the given relation, we have

$$\frac{1 - \tan\frac{1}{2}a\tan\frac{1}{2}b}{1 + \tan\frac{1}{2}a\tan\frac{1}{2}b} = \frac{1 - \tan^2\frac{1}{2}c}{1 + \tan^2\frac{1}{2}c}; \quad \therefore \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)} = \cos c.$$

Again (Art. 42),

$$\cos\beta = \frac{\cos a + \cos b}{2\cos c} = \cos^2\tfrac{1}{2}(a-b);$$

and

$$\cos\theta = \frac{\cos a\cos c\cos\beta}{\sin c\sin\beta} = \frac{\cos a - \cos b}{2\sin c\sin\beta} = -\frac{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a+b)}{\sin c\sin\beta}.$$

Substituting, we get the required result.

2. See Ex. 3, Paper III.

3. This follows at once from Exs. 8 and 9, Art. 33. See also Ex. 29, Art. 43.

4. For $2\cos c = 2\cos a\cos b = \cos(a+b) + \cos(a-b)$;

therefore

$$\cos(a-b) = 2\cos c - \cos(a+b);$$

therefore, &c.

5. See Ex. 29, Art. 43.

6. Substitute for $\tan\frac{1}{2}a$, $\tan\frac{1}{2}b$, $\tan\frac{1}{2}c$, from Ex. 27, Art. 43.

7. Proof as *in plano.*

8. If θ and ϕ be the segments of the angle B, made by the bisector of the side b, we have

$\sin a : \sin a' = \sin\theta \sin c : \sin\phi \sin \frac{1}{2} a$. But $\sin\theta : \sin\phi = \sin a : \sin c$.

(Ex. 5, Art. 31); therefore, &c.

9. $a = 58^\circ\, 33'\, 37{\cdot}48''$, $b = 41^\circ\, 47'\, 39{\cdot}62''$, $C = 104^\circ\, 48'\, 3{\cdot}58''$.

10. $A = 52^\circ\, 12'\, 36{\cdot}7''$, $B = 65^\circ\, 41'\, 37{\cdot}2''$, $b = 58^\circ\, 36'\, 37{\cdot}8''$.

V.

1. Cf. Examples 24 and 25, Art. 43.

2. $\cos AP + \cos BP = 2 \cos \frac{1}{2} c \cos \beta$, where β is the arc joining P to the middle point of AB. But

$$2 \cos \tfrac{1}{2} c = \frac{\sin a}{\sin a'}.$$

(Ex. 8, Paper IV.) Hence

$$\sin a \cos\beta + \sin a' \cos CP = \text{const.} = \sin(a + a') \cos OP,$$

[See Art. 40.]

where O is the intersection of the bisectors of the sides of ABC. Hence OP is constant, and therefore the locus is a circle, having O for pole.

3. Apply formula (2), Art. 44.

4. This follows at once from Ex. 3.

5. Reduces to $\tan c$.

6. Proof as *in plano*.

7. By aid of Ex. 6 and Art. 44.

8. $C = A + B$.

9. $A = 91^\circ\, 55'\, 57{\cdot}8''$, $B = 63^\circ\, 48'\, 41{\cdot}3''$, $c = 75^\circ\, 33'\, 8''$.

10. $a = 15^\circ\, 56'\, 30''$, $b = 46^\circ\, 42'\, 12{\cdot}4''$, $B = 75^\circ\, 28'\, 35''$.

VI.

1. If 2θ be the arc of the great circle joining the two places, $r \sin\theta = r' \sin l =$ half chord of arc, r', being the radius of the parallel of latitude, is equal to $r\cos\lambda$, where r is the radius of the earth. Hence great circle course $= 2r\theta = 2r \sin^{-1}(\cos\lambda \sin l)$; but the parallel course $= 2r'l = 2rl\cos\lambda$. Hence difference $= 2r[l\cos\lambda - \sin^{-1}(\cos\lambda \sin l)]$.

2. $\sin\frac{1}{2}a : \sin\frac{1}{2}b$ as the chords of the sides a and b. Hence we are given the base and the ratio of the sides of the chordal triangle, and the locus of *its* vertex is therefore a sphere, and hence (Art. 10) the locus of the vertex of the *spherical* triangle is a circle.

3. See the triangle DPQ, fig. 38, Art. 104.

4. See same figure.

5. If the sides CA and CB be produced to meet the circles again in X and Y, $CX = CY = 90°$; $\therefore$ C is the pole of XY; therefore, &c.

6. See Ex. 6, Paper V.

8. Use the relations given in Examples 1 and 2, Art. 31.

9. $a = 81° 46' 6{\cdot}7''$, $b = 57° 43' 43{\cdot}76''$, $C = 82° 56' 4{\cdot}4''$.

10. $a = 44° 53' 9{\cdot}4''$, $b = 69° 32' 55''$, $B = 75° 15' 22''$.

VII.

1. A small circle having the same poles as the fixed circle, and a spherical radius equal to ω.

2. Apply the relation of Paper V., Ex. 1, to a quadrilateral $ABCP$, where P is inside the triangle P; and it follows that if the angle between AP and BC is right, and also that between BP and AC, then the angle between CP and AB will be right also.

3. See Ex. 2, Paper V., and apply the theorem of Art. 40.

4. Reduces to the expression of Art. 42.

5. $$2\beta = \sqrt{a^2 + b^2 + 2ab\cos C};$$

or the corresponding equation, Ex. 14, p. 52.

6. Proof as *in plano*.

7. Let a, b, c, and therefore A, B, C, be in descending order of magnitude; then

$$\tan\tfrac{1}{2}(A + B) = \frac{\cos\frac{1}{2}(a - b)}{\cos\frac{1}{2}(a + b)}\cot\tfrac{1}{2}C;$$

and as $\cos\frac{1}{2}(a - b)$ and $\cos\frac{1}{2}C$ are essentially positive, $\tan\frac{1}{2}(A + B)$ and $\cos\frac{1}{2}(a + b)$ have the same sign; therefore, &c.

8. These relations are easily deduced from Napier's Analogies.

9. $A = 74^\circ\ 2'\ 53{\cdot}1''$, $B = 56^\circ\ 11'\ 54{\cdot}9''$, $c = 74^\circ\ 41'\ 4''$.

10. $A = 40^\circ\ 39'\ 50{\cdot}5''$, $B = 76^\circ\ 17'\ 19{\cdot}6''$, $b = 68^\circ\ 40'\ 9{\cdot}2''$.

VIII.

1. By aid of the property, Art. 25 (*a*).

$$\sin \tfrac{1}{2} a \sin \tfrac{1}{2} AP + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} BP = \sin \tfrac{1}{2} a \sin \tfrac{1}{2} CP,$$

where a is the side of the equilateral triangle; therefore, &c.

2. Expanding the terms on the right-hand side of the equation, and neglecting the powers of the radius higher than the second; remembering also that

$$\frac{1}{1-n} = (1-n)^{-1} = 1 + n, \text{ nearly},$$

if n be very small compared with unity.

Hence $$\cos^2 \tfrac{1}{2} A = \frac{s(s-a)}{bc} \left[1 - \frac{s^2 + (s-a)^2 - b^2 - c^2}{6r^2}\right]$$

$$= \frac{s(s-a)}{bc} \left[1 - \frac{(s-b)(s-c)}{3r^2}\right] = \cos^2 \tfrac{1}{2} A' - \frac{\Delta^2}{3bcr^2}.$$

Again, let $\tfrac{1}{2} A = \tfrac{1}{2} A' + \theta$; then $\cos^2 \tfrac{1}{2} A = \cos^2 \tfrac{1}{2} A' - \theta \sin A'$, nearly; hence

$$\theta \sin A' = \frac{\Delta^2}{3bcr^2}, \text{ i.e. } 2\theta = \frac{\Delta}{3r^2};$$

but 2θ is the difference between A and A'; therefore, &c.

(Cf. Art. 29.)

3. The method in either case is similar to that given in the preceding Example. See also Art. 29.

4. Squaring the expression for $\sin \tfrac{1}{2} a$, given in Art. 33, and dividing by the product of the analogous expressions for $\sin \tfrac{1}{2} b$ and $\sin \tfrac{1}{2} c$, since $\cos(S-A)$, $\cos(S-B)$, $\cos(S-C)$ become *in plano*, respectively, $\sin A$, $\sin B$, $\sin C$, we get

$$\frac{a^2}{c} = \frac{\sin^2 A}{\sin B \sin C}.$$

Hence $$\frac{a^2 \sin B \sin C}{\sin(B+C)} = bc \sin A = 2\Delta.$$

5. See Ex. 4.

6. See Ex. 3.

7. Draw the required figure, and join AA'. Denote the angles BAA' and $BA'A$, respectively, by x and x'. Then we have—

$$\begin{aligned}
&(1)\ \cos B = -\cos x \cos x' + \sin x \sin x' \cos AA';\\
&(2)\ \cos B' = -\cos(A - x')\cos(A' - x) + \sin(A - x)\sin(A' - x')\cos AA';\\
&(3)\ \cos C = \cos x \cos(A' - x') + \sin x \sin(A' - x')\cos AA';\\
&(4)\ \cos C' = \cos x' \cos(A - x) + \sin x' \sin(A - x)\cos AA'.
\end{aligned}$$

Multiply equations (1) and (2), and subtract from the result the product of equations (3) and (4); therefore, &c.

8. (α) follows at once from Ex. 7. (β) We have

$$\Sigma(\cos B \cos B' - \cos C \cos C') = \Sigma \sin A \sin A'; \text{ therefore, \&c.}$$

9. $A = 94° 56' 36{\cdot}6''$, $B = 63° 0' 24{\cdot}6''$, $C = 74° 6' 20{\cdot}6''$.

10. $A = 54° 47' 53{\cdot}6''$, $b = 34° 36' 11''$, $c = 50° 7' 13{\cdot}2''$.

IX.

1. By the aid of Paper VIII., Ex. 7, we obtain

$$\Sigma(\sin A \sin A' \cos \delta_1) = \Sigma(\cos B \cos B' - \cos C \cos C') = 0.$$

2. It is evident, in the preceding Example, that if δ_1 and δ_2 are quadrants, each term in that equation vanishes. Hence $\cos \delta_3 = 0$; therefore, &c.

3. The great circle on which they lie is the polar of the intersection of the perpendiculars of the triangle. (See Note, Art. 21.)

4. This follows from Paper V., Ex. 1. See also the preceding solution, Ex. 3.

5. An obvious particular case of the preceding Example, and which may be verified directly as follows:—Draw perpendiculars from A and P to the base BC; the distance between their feet is $AP \cos X$; then

$$\Sigma\, a\, AP \cos X = 2\,\Sigma(BP^2 - CP^2) - 2\,\Sigma(b^2 - c^2):$$

i. e.
$$a\,AP \cos X + b\,BP \cos Y + c\,CP \cos Z = 0.$$

6. In Ex. 4, let $a = b = c$; therefore, &c.

7. The left-hand side of the equation reduces to

$$\cos a - \cos\beta \cos\gamma \cos^2 p - \sin b \sin c \cos A; \text{ since } \beta + \gamma = a,$$

and
$$\sin b \sin c \sin A = \sin a \sin p.$$

Again, $\cos a - \sin b \sin c \cos A = \cos b \cos c.$

Hence the given expression becomes

$$\cos b \cos c - \cos\beta \cos\gamma \cos^2 p, \text{ or } \cos\beta \cos\gamma \cos^2 p - \cos\beta \cos\gamma \cos^2 p.$$

[Art. 44 (1).]

Again, $$\cot A = \frac{\cos a - \cos\beta \cos\gamma \cos^2 p}{\sin a \sin p}.$$

Now A is a maximum when $\cot A$ is a minimum; *i.e.* when $\cos\beta \cos\gamma$ is a maximum; but

$$2\cos\beta\cos\gamma = \cos(\beta+\gamma) + \cos(\beta-\gamma) = \cos a + \cos(\beta-\gamma).$$

From this result it follows that the vertical angle is a maximum when the perpendicular from it on the base bisects the base; therefore, &c.

[Cf. TOWNSEND'S *Mod. Geom.*, vol. i., Ex. 7, p. 47.]

8. For $\sin\frac{1}{2}a$, $\sin\frac{1}{2}b$, &c., substitute the chords of the arcs a, b, &c., to which they are proportional, and the expression reduces to that which gives a diagonal of a plane cyclic quadrilateral in terms of its sides.

[See TODHUNTER'S *Plane Trig.*, Art. 254.]

9. $a = 66^\circ\ 19'\ 44{\cdot}17''$, $b = 45^\circ\ 6'\ 32{\cdot}75''$, $C = 94^\circ\ 21'\ 40{\cdot}3''$.

10. $A = 51^\circ\ 3'\ 4{\cdot}2''$, $b = 55^\circ\ 20'\ 23{\cdot}35''$, $c = 66^\circ\ 30'\ 30{\cdot}6''$.

X.

1.* $A+a=\alpha$, $B+b=\beta$, $C+c=\gamma$, $A-a=\theta$, $B-b=\phi$, $C-c=\psi$;

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}; \ \therefore\ \frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\alpha} = \frac{\tan\frac{1}{2}\phi}{\tan\frac{1}{2}\beta} = \frac{\tan\frac{1}{2}\psi}{\tan\frac{1}{2}\gamma}; \qquad (1)$$

but $$\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(a+b)} = \frac{\sin\frac{1}{2}C}{\cos\frac{1}{2}c} = \frac{\cos\frac{1}{2}(\pi - C)}{\cos\frac{1}{2}c}; \qquad \text{(GAUSS.)}$$

therefore

$$\tan\tfrac{1}{4}(\theta+\phi)\tan\tfrac{1}{4}(\beta+\alpha) = \tan\tfrac{1}{4}(\pi-\gamma)\tan\tfrac{1}{4}(\pi-\psi). \qquad (2)$$

Again,

$$\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(a-b)} = \frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}c} = \frac{\sin\frac{1}{2}(\pi-C)}{\sin\frac{1}{2}c}; \ \therefore\ \frac{\tan\frac{1}{4}(\theta-\phi)}{\tan\frac{1}{4}(\alpha-\beta)} = \frac{\tan\frac{1}{4}(\pi-\gamma)}{\tan\frac{1}{4}(\pi-\psi)}. \qquad (3)$$

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* We are indebted to Mr. T. A. Finch for the first solution we give of this question; and to Mr. Robert Russell for an independent method. Mr. Russell remarks that the equations used by him to solve the triangle are likewise applicable to the more general problem—"Given *any three* of the six quantities $A \pm a$, $B \pm b$, $C \pm c$; solve the triangle."

(2) and (3) multiplied together give the ratio $\cos\frac{1}{2}\theta : \cos\frac{1}{2}\phi$, and from (1) we have the ratio $\tan\frac{1}{2}\theta : \tan\frac{1}{2}\phi$; and hence θ, ϕ, ψ are known; therefore, &c.

Cor.—Given $A - a$, $B - b$, and $C - c$; find the remaining parts.

Otherwise thus:—

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}; \text{ therefore } \frac{\sin A + \sin a}{\sin A - \sin a} = \ldots = \ldots$$

Hence
$$\frac{\tan\frac{1}{2}(A+a)}{\tan\frac{1}{2}(A-a)} = \frac{\tan\frac{1}{2}(B+b)}{\tan\frac{1}{2}(B-b)} = \ldots$$

Let
$$\cot\tfrac{1}{2}(A+a) = \iota \sin X,$$
and
$$\cot\tfrac{1}{2}(A-a) = \iota \sin x, \text{ where } \iota = \sqrt{-1}.$$

Hence
$$\frac{\sin X}{\sin x} = \frac{\sin Y}{\sin y} = \frac{\sin Z}{\sin z};$$

from which the question reduces to—Having given the three angles X, Y, Z of a spherical triangle; find the sides.*

We have thus—
$$\cos x = \frac{\cos X + \cos Y \cos Z}{\sin Y \sin Z};$$

or
$$\operatorname{cosec}\tfrac{1}{2}(A-a) = \frac{\operatorname{cosec}\frac{1}{2}(A+a) + \operatorname{cosec}\frac{1}{2}(B+b)\operatorname{cosec}\frac{1}{2}(C+c)}{-\cot\frac{1}{2}(B+b)\cot\frac{1}{2}(C+c)};$$

or
$$\sin\tfrac{1}{2}(A-a) = -\frac{\sin\frac{1}{2}(A+a) \quad \cos\frac{1}{2}(B+b)\cos\frac{1}{2}(C+c)}{\sin\frac{1}{2}(A+a) + \sin\frac{1}{2}(B+b)\sin\frac{1}{2}(C+c)}.$$

2. Let $AO = BO = CO = R$, and let $PO = \delta$, and the angle $POC = \theta$. Then we have—

$$\cos AP = \cos R \cos\delta + \sin R \sin\delta \cos(120 - \theta). \qquad (1)$$

$$\cos BP = \cos R \cos\delta + \sin R \sin\delta \cos(120 + \theta). \qquad (2)$$

$$\cos CP = \cos R \cos\delta + \sin R \sin\delta \cos\theta. \qquad (3)$$

Hence, by addition,

$$\cos AP + \cos BP + \cos CP = 3\cos R \cos\delta. \qquad (4)$$

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* Mr. Robert Russell, *Quarterly Journal of Mathematics*, vol. xx.

Multiply (2) by ω and (3) by ω^2, where $\omega^3 = 1$, and add; then

$$\cos AP + \omega \cos BP + \omega^2 \cos CP$$

$$= \sin R \sin \delta \left\{\cos (120 - \theta) + \omega \cos (120 + \theta) + \omega^2 \cos \theta\right\}. \qquad (5)$$

Similarly,

$$\cos AP + \omega^2 \cos BP + \omega \cos CP$$

$$= \sin R \sin \delta \left\{\cos (120 - \theta) + \omega^2 \cos (120 + \theta) + \omega \cos \theta\right\}. \qquad (6)$$

Multiplying equations (5) and (6), we have, on reduction,

$$\cos^2 AP + \cos^2 BP + \cos^2 CP - \cos BP \cos CP - \ldots = \tfrac{9}{4} \sin^2 R \sin^2 \delta;$$

or by the aid of (4) $= \frac{1}{4} (\cos AP + \cos BP + \cos CP)^2 \tan^2 R \tan^2 \delta$.

NOTE.—To simplify the right-hand side of the product of (5) and (6), throw the part within the brackets into the determinant form—

$$\begin{vmatrix} 1 & \cos \theta & \cos (120 + \theta) \\ 1 & \cos (120 - \theta) & \cos \theta \\ 1 & \cos (120 + \theta) & \cos (120 - \theta) \end{vmatrix};$$

and remembering that

$$\cos \theta + \cos (120 + \theta) + \cos (120 - \theta) = 0,$$

the reduction presents no difficulty.

(*Educational Times.*)

3. The right angles on the primitive figure are transformed into the diagonals of the supplemental figure; therefore, &c.

4. It has been seen in Paper X., Ex. 2, that

$$\cos AP + \cos BP + \cos CP = 3 \cos AO \cos OP.$$

In this instance, $\cos AO = \dfrac{1}{\sqrt{3}}$;

therefore, &c.

5. $\cos A = \dfrac{\tan b}{\tan c}$; therefore $\dfrac{\tan c + \tan b}{\tan c - \tan b} = \dfrac{\sin (c + b)}{\sin (c - b)} =$ given.

Hence if $c + b$ be known, $c - b$ is determined, and therefore c and b.

6. See fig. 10. The arc $LM = 2\,XY$. Hence the angle AOB is known, and consequently the triangle AOB is completely determined; and since $OA + OC = \pi$, the locus of the vertex is a circle, of which O is the pole.

7. This follows at once from Ex. 5, where it is shown that $\sin (c + b)$ varies as $\sin (c - b)$ when A is given.

8. From A, one of the extremities of the given base, let fall AP perpendicular to the given great circle, and produce AP to Q, so that $AP = PQ$. Join Q to B, the other extremity of the given base. Let BQ cut the given circle in C. Then ABC and its colunar ABC' are the required triangles. C' being diametrically opposite to C, is on the given circle.

9. $A = 70° \; 25' \; 33\tfrac{1}{2}''$, $B = 51° \; 5' \; 47''$, $c = 73° \; 19' \; 28\tfrac{1}{2}$.

10. $A = 70° \; 22' \; 16{\cdot}1''$, $B = 49° \; 25' \; 12{\cdot}3''$, $c = 61° \; 54' \; 6''$.

XI.

1. Form a spherical triangle by joining the poles of the given great circles to the point P, and it is obvious that of this triangle we are given the base and the sum of the cosines of the sides; therefore, &c. (See Art. 42.)

2. By the method of the previous exercise. (See also Ex. 1, Art. 42.)

3. See Paper VII., Ex. 3.

4. We have $a = R(\cos \alpha + \cos \delta)$,
and $c = R(\cos \alpha - \cos \delta)$;
therefore $a + c = 2R \cos \alpha = 0$, if $\alpha = 90°$.

5. From the extremities A and B of the base let fall secondaries AX and BY on the given locus. Then
$\cos c = \cos a \cos b = \cos AX \cos CX \cos BY \cos CY$; but AX, BY, and XY are given arcs. Hence it is required to divide a given arc XY at a point C, such that $\cos CX \cos CY$ may be given.

6. The product is evidently

$$\sin a \sin b \cos^2 \tfrac{1}{2} C = \frac{\sin a \sin b \sin s \sin (s - c)}{\sin a \sin b} = \sin s \sin (s - c),$$

which is given.

7. The product is

$$\sin a \sin b \sin^2 \tfrac{1}{2} C = \sin (s - a) \sin (s - b);$$

therefore, &c.

8. $\sin p_1 \sin p_2 = \sin a \sin b \cos^2 \tfrac{1}{2} C$; $\sin p_3 \sin p_4 = \sin a \sin b \sin^2 \tfrac{1}{2} C$; therefore, &c. (See Exs. 6 and 7.)

9. $a = 29° \; 26' \; 30''$, $b = 84° \; 37' \; 15''$, $c = 85° \; 19' \; 2''$.

10. $A = 82° \; 24' \; 33{\cdot}4''$ $B = 45° \; 2' \; 9{\cdot}88''$, $c = 60° \; 20' \; 10{\cdot}5''$.

XII.

1. For by Art. 42 we have

$$\cos a + \cos b = 2 \cos \tfrac{1}{2} c \cos \beta,$$

$$\cos a - \cos b = 2 \sin \tfrac{1}{2} c \cos \beta' :$$

squaring each of these equations, and subtracting the results, (α) follows at once; when the results are added, we obtain (β).

2. See Art. 36 (12).

3. This equation is an expression for the base in terms of two sides and the angle included between them, and must, therefore, be identical with the form

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

The identity of these equations will be at once made evident by the substitution of

$$\frac{1 - \tan^2 \frac{1}{2} a}{1 + \tan^2 \frac{1}{2} a} \text{ for } \cos a, \text{ \&c.}$$

and

$$\frac{2 \tan \frac{1}{2} a}{1 + \tan^2 \frac{1}{2} a} \text{ for } \sin a, \text{ \&c.}$$

4. Substitute the value of $\tan \frac{1}{2} a$ in terms of the angles (Art. 36).

5. Apply Art. 40.

6. *Firstly*—By the rule of sines—

$$\frac{\sin a}{\sin b} = \frac{\sin (c - \alpha)}{\sin \alpha} = \sin c \cot \alpha - \cos c.$$

Hence

$$\cot \alpha = \frac{\sin b \cos c + \sin a}{\sin b \sin c};$$

i.e.

$$\sin^2 \alpha = \frac{\sin^2 b \sin^2 c}{\sin^2 a + \sin^2 b + 2 \sin a \sin b \cos c}.$$

Similarly,

$$\sin^2 \beta = \frac{\sin^2 a \sin^2 c}{\sin^2 a + \sin^2 b + 2 \sin a \sin b \cos c};$$

therefore, &c.

Secondly—

$$\sin^2 a \sin^2 \theta = \sin^2 b \sin^2 \tfrac{1}{2} C.$$

Substitute the value already found for $\sin^2 a$; therefore, &c.

Thirdly—Multiply together the results in Exs. 9 and 10, Art. 35.

7. When $c - b$ is a maximum (by Paper x. Ex. 7), $c = 90 - b$; hence

$$\cos c = \sin b = \cos a \cos b, \quad \textit{i.e.} \quad \cos a = \tan b = \cot c.$$

Again, $\cos A = \tan b \cot c$, and is therefore equal to $\cos^2 a$.

8. $a = 53^\circ\ 24'\ 11{\cdot}78''$, $b = 37^\circ\ 27'\ 55{\cdot}34''$, $C = 76^\circ\ 47'\ 34{\cdot}84'$.

9. $a = 36^\circ\ 9'\ 12{\cdot}78''$, $b = 44^\circ\ 23'\ 39{\cdot}54''$, $c = 40^\circ\ 0'\ 3{\cdot}8''$.

10. $A = 94^\circ\ 16'\ 17\tfrac{1}{4}''$, $B = 56^\circ\ 55'\ 38\tfrac{3}{4}''$, $c = 73^\circ\ 30'\ 28''$.

11. $b = 42^\circ\ 37'\ 43\tfrac{1}{4}''$, $c = 56^\circ\ 7'\ 29\tfrac{1}{2}''$, $B = 54^\circ\ 39'\ 26{\cdot}4''$,

or, $b = 137^\circ\ 22'\ 16\tfrac{3}{4}''$, $c = 123^\circ\ 52'\ 30\tfrac{1}{2}''$, $B = 125^\circ\ 20'\ 33{\cdot}6''$

12. $b = 41^\circ\ 12'\ 54\tfrac{1}{2}''$, $c = 61^\circ\ 57'\ 58\tfrac{1}{2}''$, $B = 48^\circ\ 17'\ 8\tfrac{1}{4}''$,

or, $b = 138^\circ\ 47'\ 5\tfrac{1}{2}''$, $c = 118^\circ\ 2'\ 1\tfrac{1}{2}''$, $B = 131^\circ\ 42'\ 51\tfrac{3}{4}''$.

13. $a = 82^\circ\ 38'\ 40{\cdot}8''$, $b = 67^\circ\ 6'\ 44{\cdot}48''$, $C = 88^\circ\ 58'\ 12{\cdot}38''$.

14. $a = 71^\circ\ 35'\ 16''$, $b = 40^\circ\ 33'\ 12''$, $C = 76^\circ\ 2'\ 39{\cdot}72''$.

15. $a = 57^\circ\ 7'\ 7{\cdot}42''$, $b = 69^\circ\ 22'\ 20{\cdot}4''$, $c = 54^\circ\ 3'\ 18{\cdot}2''$.

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BY

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AND

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